Astroid *

1 History

Quote from Robert C. Yates, 1952:

> The cycloidal curves, including the astroid, were discovered by Roemer (1674) in his search for the best form for gear teeth. Double generation was first noticed by Daniel Bernoulli in 1725.

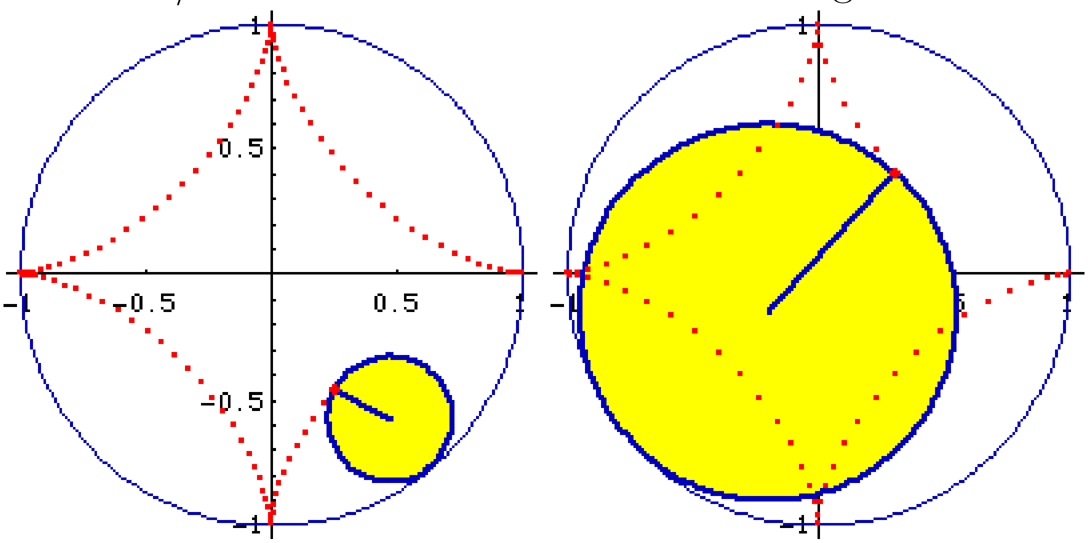
Quote from E. H. Lockwood, 1961:

> The astroid seems to have acquired its present name only in 1838, in a book published in Vienna; it went, even after that time, under various other names, such as cubocycloid, paracycle, four-cusp-curve, and so on. The equation $x^{2/3} + y^{2/3} = a^{2/3}$ can, however, be found in Leibniz's correspondence as early as 1715.

*This file is from the 3D-XploreMath project.
Please see http://rsp.math.brandeis.edu/3D-XplorMath/index.html

2 Description

Astroid is a special case of hypotrochoid. Astroid is defined as the trace of a point on a circle of radius r rolling inside a fixed circle of radius $4r$ or $4/3r$. The latter is known as double generation.



3 Formulas

Parametric: $(\cos^3(t), \sin^3(t))$, $0 < t \leq 2\pi$. This formula gives an astroid centered on the origin with one cusp at (1,0).

Cartesian equation: $x^{2/3}+y^{2/3} = 1$ or $(x^2+y^2-1)^3+27x^2y^2 = 0$.

To derive the Cartesian equation, let

$$x = cos^3 t$$
$$y = sin^3 t$$

Raise both sides to the power of 2/3 we have

$$x^{2/3} = \cos^2 t$$
$$y^{2/3} = \cos^2 t$$

(note that we do not have to worry about the sign in squaring both sides, because $x := \cos^3 t$ is a definition. It is not an equation with an unknown) Add the two equations together we have

$$x^{2/3} + y^{2/3} = \cos^2 t + \sin^2 t = 1$$

It turns out that we can get rid of the fractional power. Raise both sides by powers of 3:

$$\begin{aligned} x^2 + y^2 + 3x^{4/3}y^{2/3} + 3x^{2/3}y^{4/3} &= 1 \\ 3(x^{4/3}y^{2/3} + x^{2/3}y^{4/3}) &= 1 - x^2 - y^2 \end{aligned}$$

replace $x^{2/3}$ by $1 - y^{2/3}$ and $y^{2/3}$ by $1 - x^{2/3}$ to obtain

$$\begin{aligned} 3(x^{4/3}(1 - x^{2/3}) + (1 - y^{2/3})y^{4/3}) &= 1 - x^2 - y^2 \\ 3(x^{4/3} + y^{4/3}) &= 1 + 2(x^2 + y^2) \end{aligned}$$

Now since $x^{2/3} + y^{2/3} = 1$, square both sides and simplify we have $x^{4/3} + y^{4/3} = 1 - 2x^{2/3}y^{2/3}$. We now replace $x^{4/3} + y^{4/3}$ in the above equation to get

$$\begin{aligned} 3(1-2x^{2/3}y^{2/3}) &= 1+2(x^2+y^2) \\ -6x^{2/3}y^{2/3} &= 1+2(x^2+y^2)-3 \end{aligned}$$

Now raise both sides by 3 and we arrive:

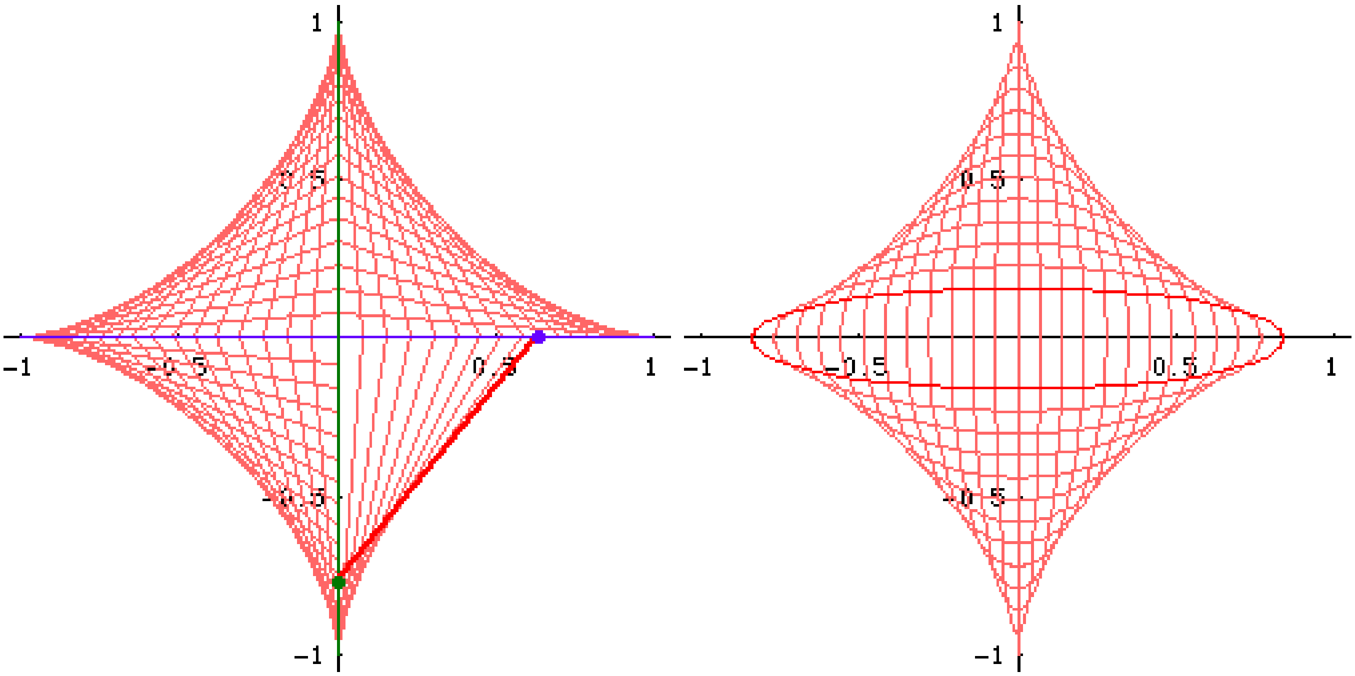
$$-6^3x^2y^2 = (1+2(x^2+y^2)-3)^3$$

4 Properties

4.1 Trammel of Archimedes and Envelope of Ellipses

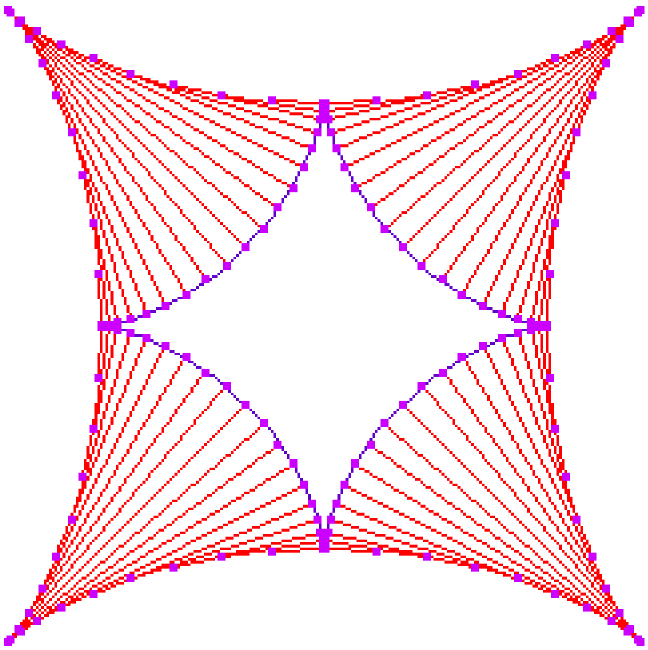
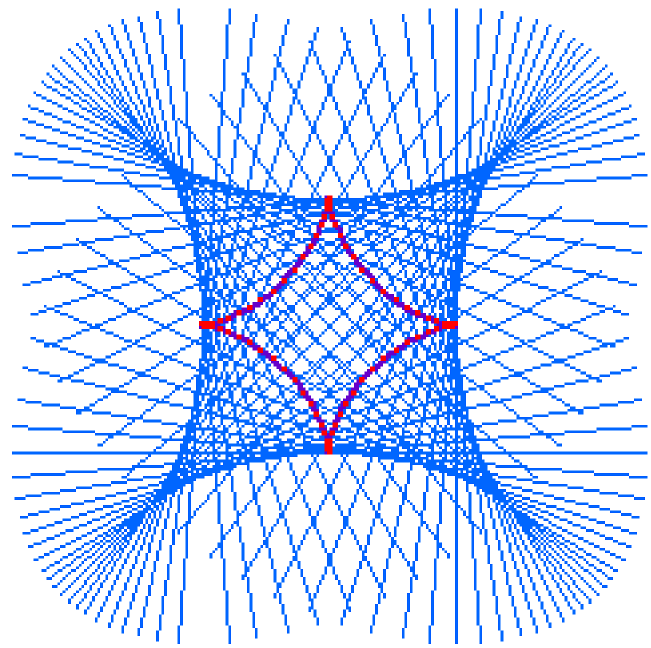
Define the axes of the astroid to be the two perpendicular lines passing its cusps. Property: The length of tangent cut by the axes is constant. A mechanical devise where a fixed bar with endings sliding on two penpendicular tracks is called the Trammel of Archimedes. The envelope of the moving bar is then the astroid. A fixed point on the bar will trace out an ellipse.

Astroid is also the envelope of co-axial ellipses whose sum of major and minor axes is contsant.



4.2 Evolute of Astroid

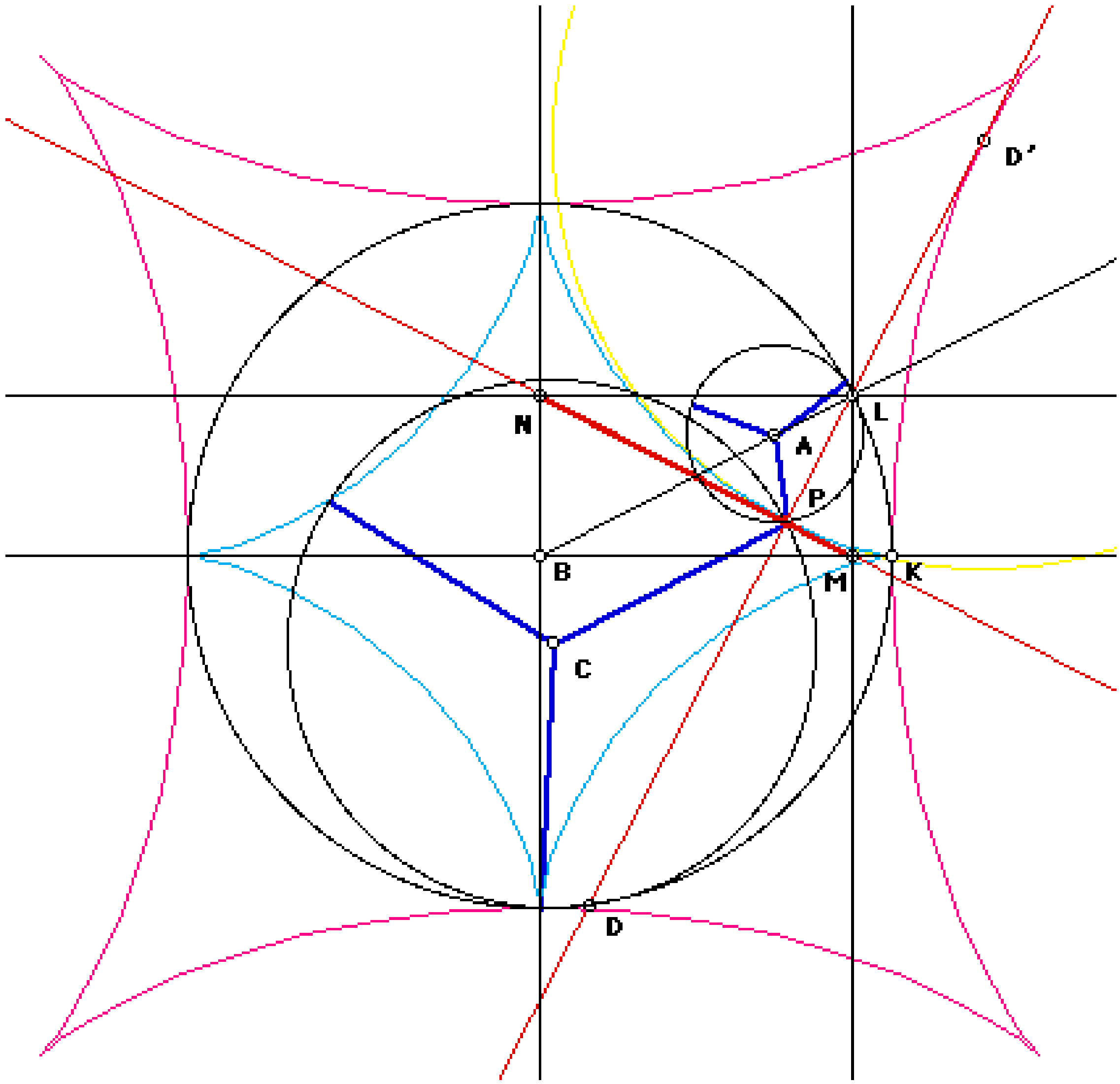
The evolute of an astroid is another astroid. (all epi/hypocycloids' evolute is equal to themselves) In the first figure below, points on the curve are connected to their center of osculating circles. in the second figure below, the evolute is drawn as the envelope of normals.

4.3 Curve Construction

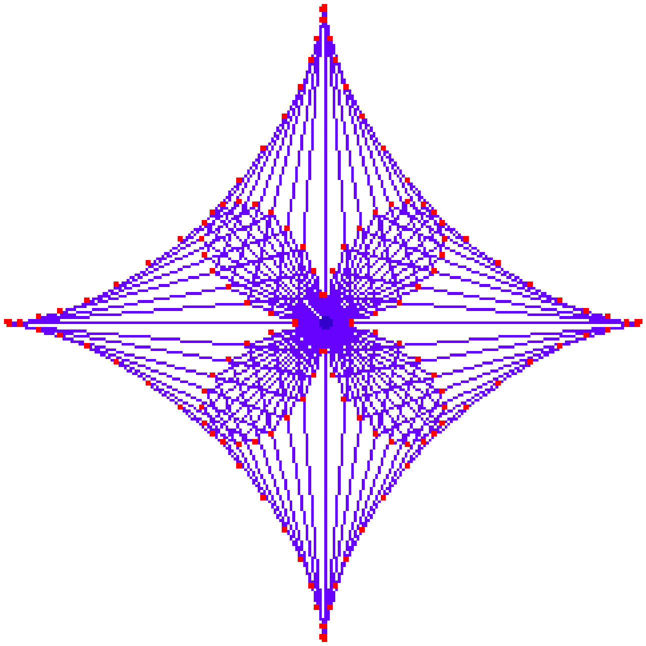
The astroid is rich in properties that one can construct the curve, its tangent, and center of osculating circle, and device other mechanical ways to generate the curve.

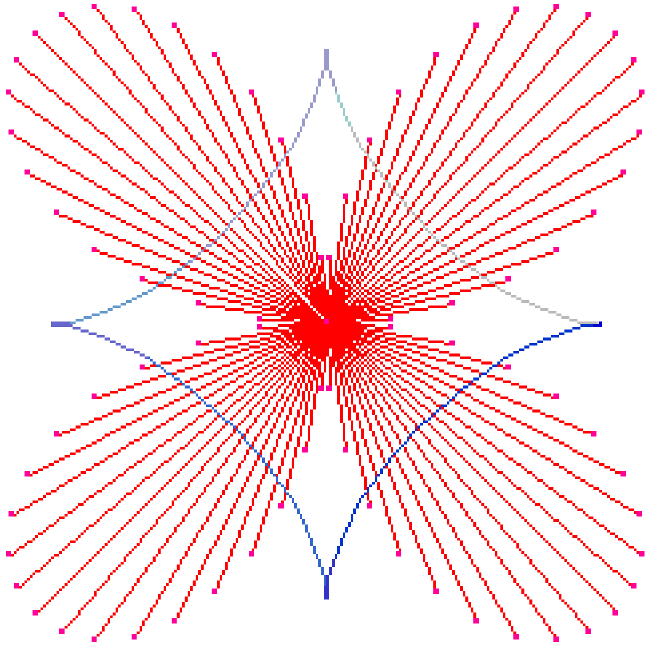
Let there be a circle centered on B passing K. We will construct an astroid centered on B with one cusp at K. Let B be the origin, and K be the ponit (1,0). Let L be a point on circle$[B, BK]$. Drop a line from L perpendicular to x-axis, let M be their intersection. Similarly drop a line from L perpendicular to y-axis, call the intersection N. Let P be a point on MN such that LP and MN are perpendicular. Now, P is a point on the astroid, and MN is its tangent, LP is its normal. Let D be the intersection of LP and circle$[B, BK]$. Let D' be the reflection of D thru MN. Now, D' is the center of osculating circle at P.



4.4 Redal, Radial, and Rose

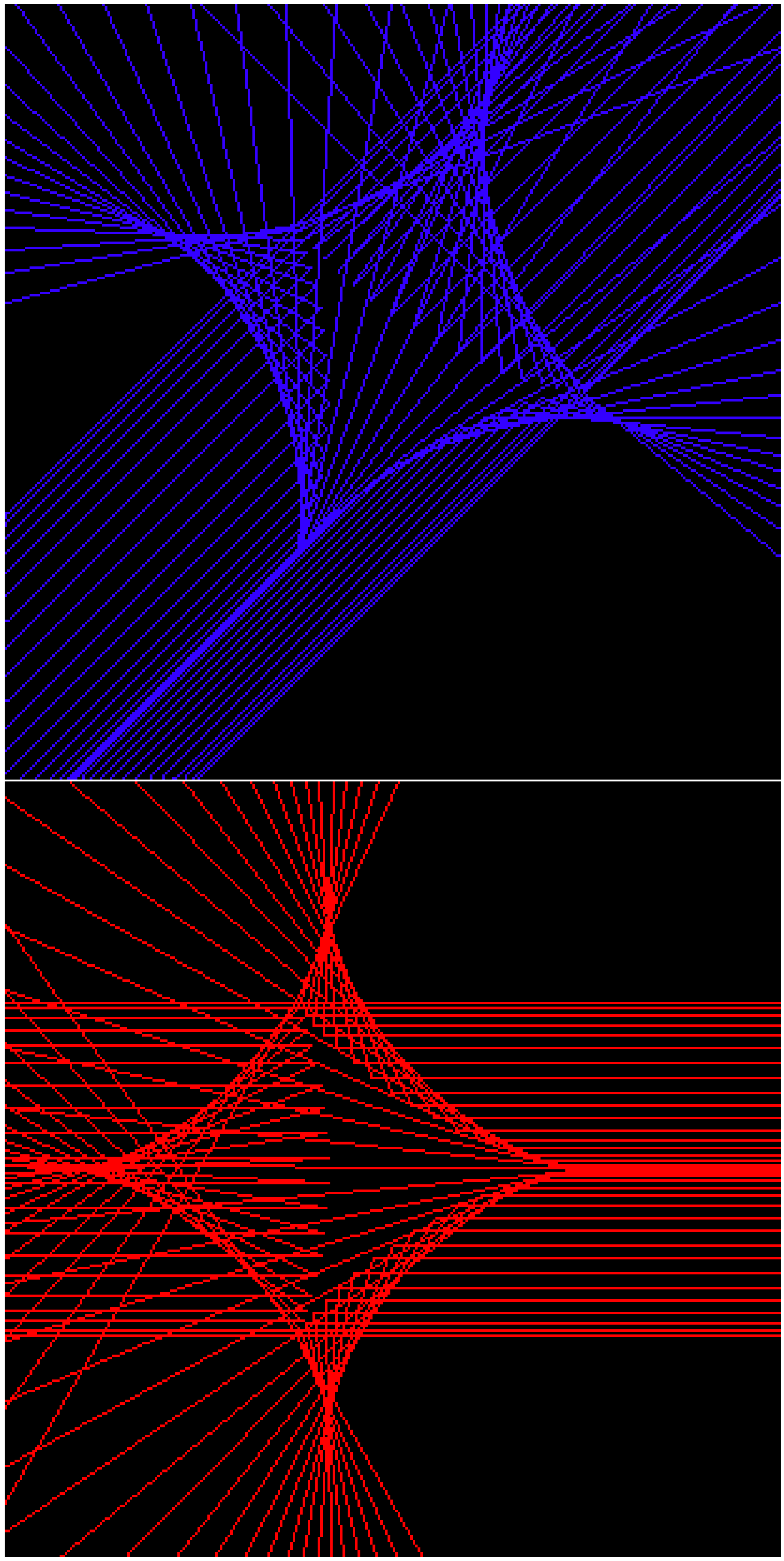
The pedal of an astroid with respect to its center is 4 petalled rose, called a quadrifolium. Astroid's radial is also quadrifolium. (all epi/hypocycloid's pedal and radial are equal, and they are roses.)





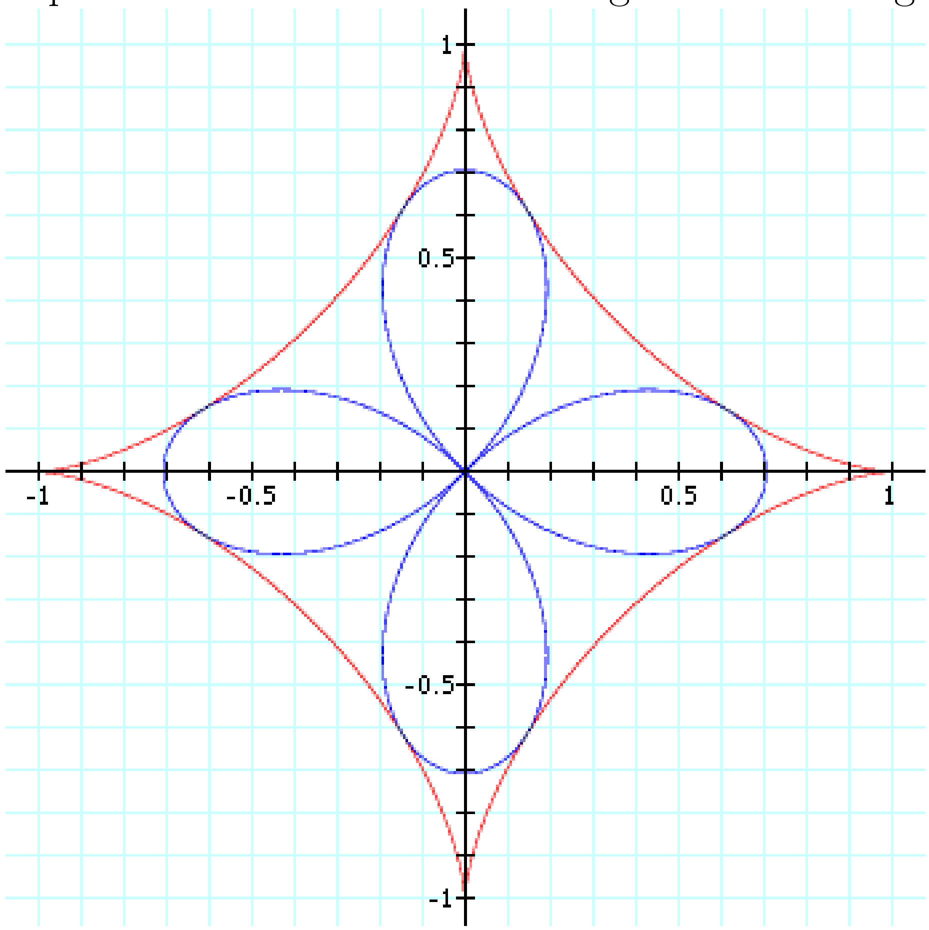
4.5 Catacaustic and Deltoid

Astroid is the catacaustic of deltoid with parallel rays in any direction.



4.6 Orthoptic

The orthoptic with respect to its center is $r^2 = (1/2)\cos(2\theta)^2$. [Robert C. Yates.] Recall that an orthoptic of a curve C is the locus of all points P where the curve's tangents meet at right angles.



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