

# The reflection of light rays in a cup of coffee or the curves obtained with $b^n \pmod p$

by Simon Plouffe

based on works done in the years 1974-79

Take a circle centered at 0,0, divide it into  $p$  parts and take  $2^n \pmod p$ , if 2 is a primitive root of  $p$  then you will have this nice drawing of a cardioid. That same figure can be obtained by a source at (1,0) that projects  $p$  rays at the  $p$  equally spaced points on the circumference. If the rays are reflected once then we obtain the curve. You may obtain the same curve by looking at a cup of coffee when you are under the sun during day.

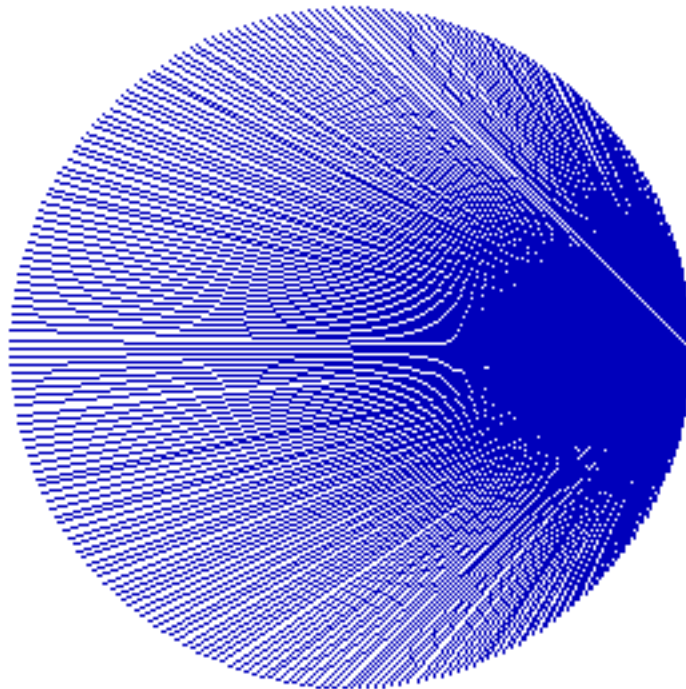


Fig 0. First rays hitting the cup of coffee.

The second figure is a cardioid and can also be generated with a circle divided into  $p$  (prime) parts and by taking  $2^n \pmod p$ , 2 being a primitive root of  $p$  (so that there are  $p-1$  residues). For doing the figure take the successive residues  $(\pmod p)$  and join them with a line.

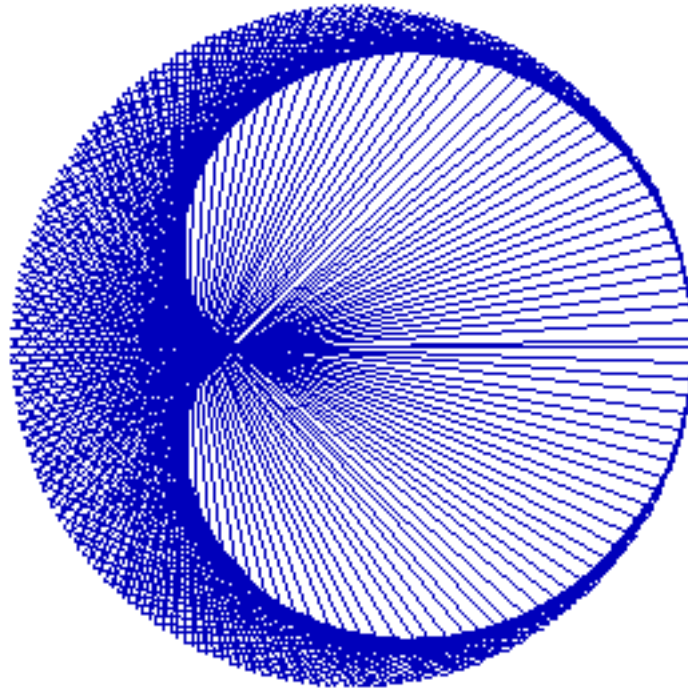


Fig 1. The first reflection is equivalent to  $2^n \bmod p$  when 2 is a primitive root.

By plotting the residues mod  $p$  joined by lines it is the same as representing  $1/p$  in base 2 by considering the binary expansion of that number (which has a period of  $p-1$  since 2 is a primitive root). For this we map the number  $x$  in  $[0,1]$  to  $x \exp(2\pi i x)$

If we plot  $1/257$  in base 10 or equivalently if we plot the residues of  $10^n \bmod 257$  we obtain a strange figure with 9 cusps and many other structures AND it is also the 56th reflection of the light hitting the side of a cup of coffee... This fact (as far as I know) is not easily explained. If you ever find an explanation please **let me know!**

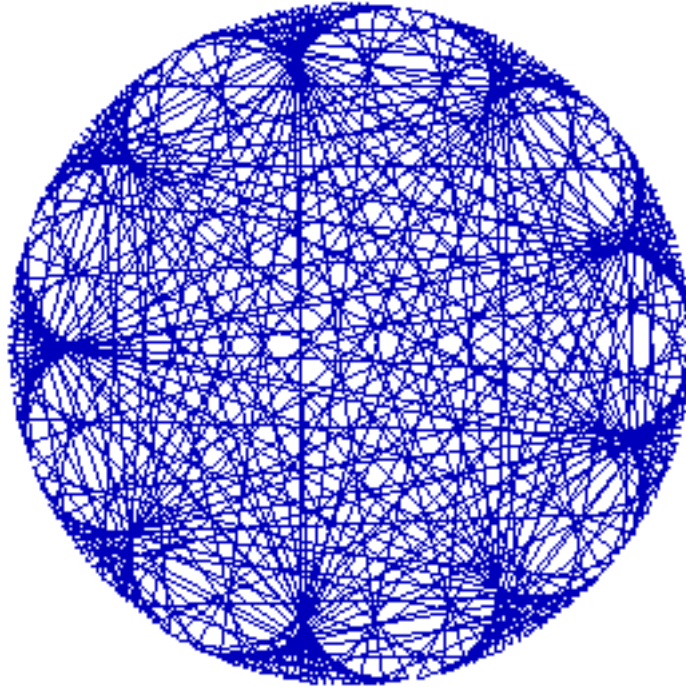


Fig 2. The 56 th reflection of light rays  
equivalent to  $10^n \bmod 257$ .

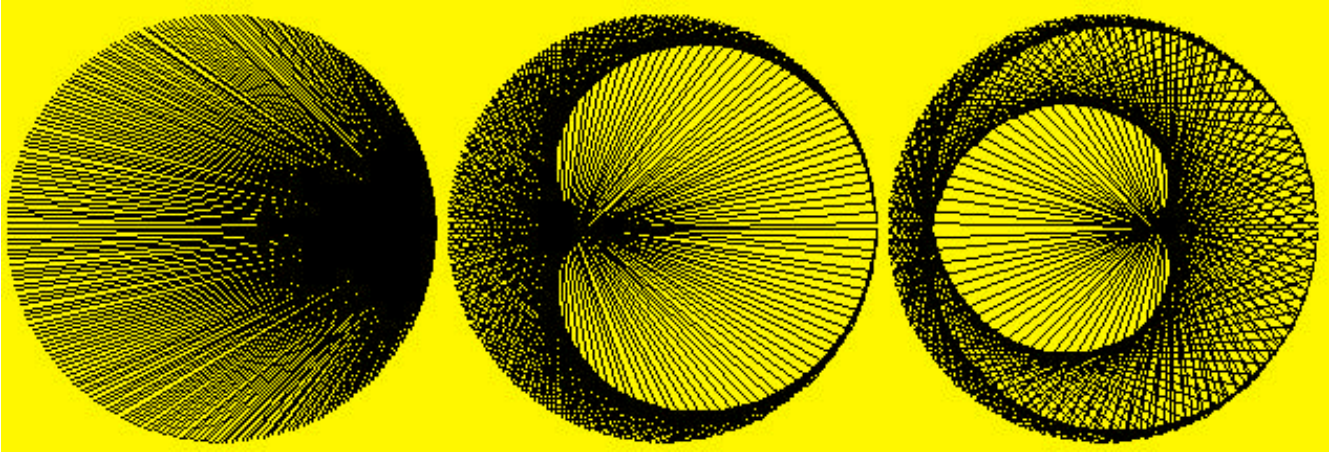
note that there are 9 cusps and  $23/2$  cusps or 23 cusps in 2 turns.

The number of different figures obtained with a prime  $p$  are  $(p-1)/2 + 1$ , the figures are eventually repeating after a number of reflections.

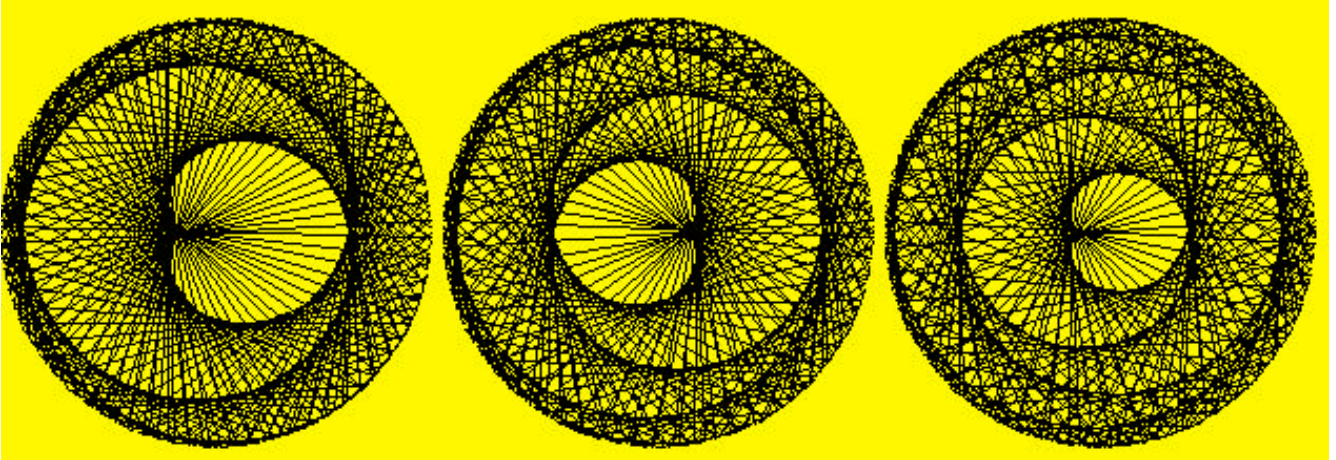
The number of principal cusps are  $b-1$  when  $b$  is a primitive root of  $p$  and  $b$  relatively small. It is difficult to come with a general formula. For  $3^n \bmod 257$  we have (as expected), that rule is no longer valid when  $b$  gets larger, I have no explanation for the general case. See below the other templates. With lots of experiments, I came with this formula, it explains many cases but not all. For any  $p$  and any  $g$  (or  $b$ ) there are no (not known to me) other formulas.

Principle 1 : The number of cusps of a drawing of  $b^n \bmod p$  will be  $b-1$ .

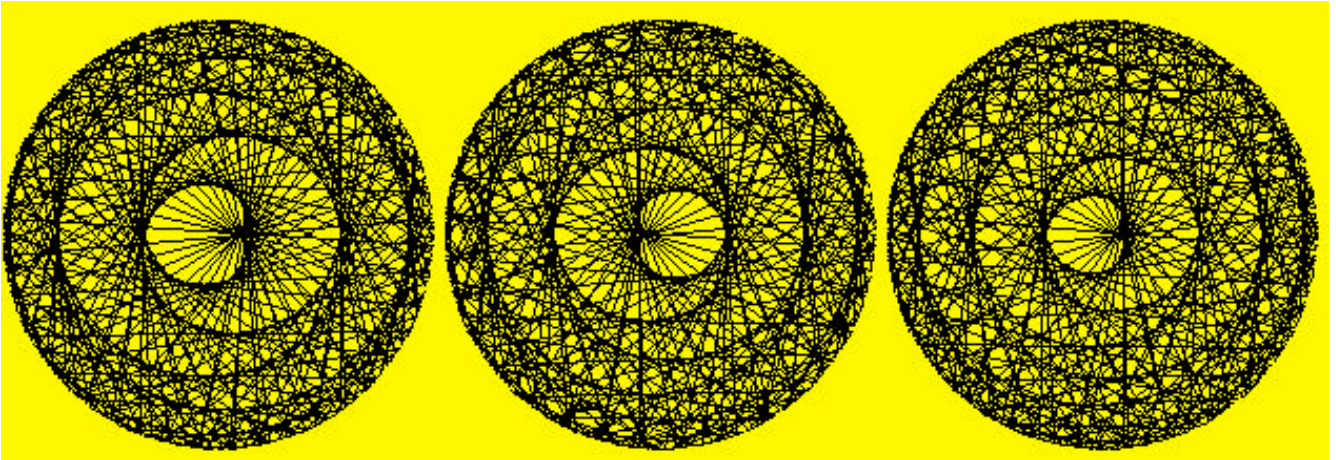




Other figures in order of reflections. 0 th reflection, 1st reflection and second reflection.

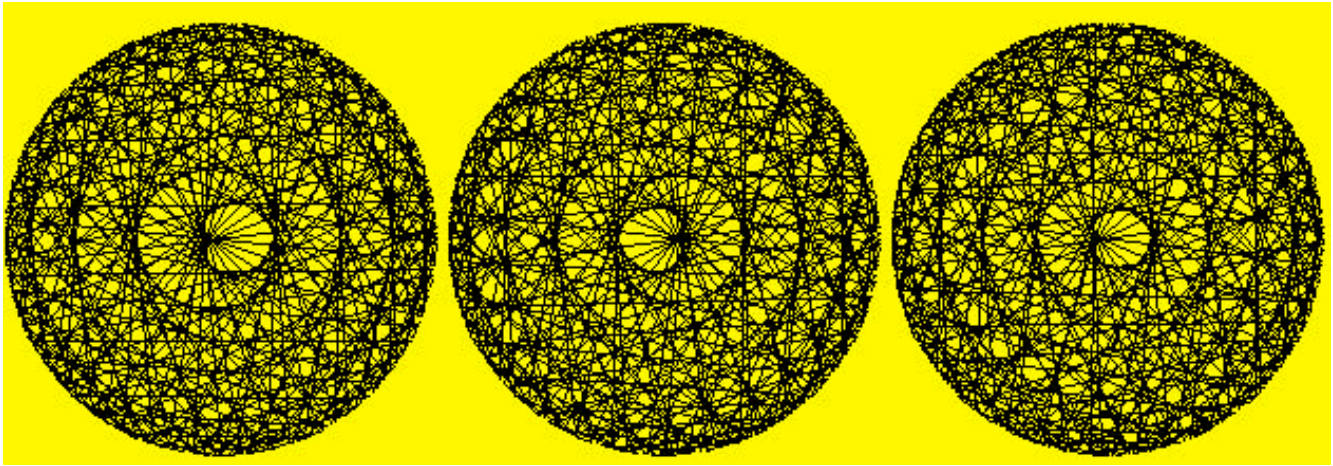


reflections 3, 4 and 5.

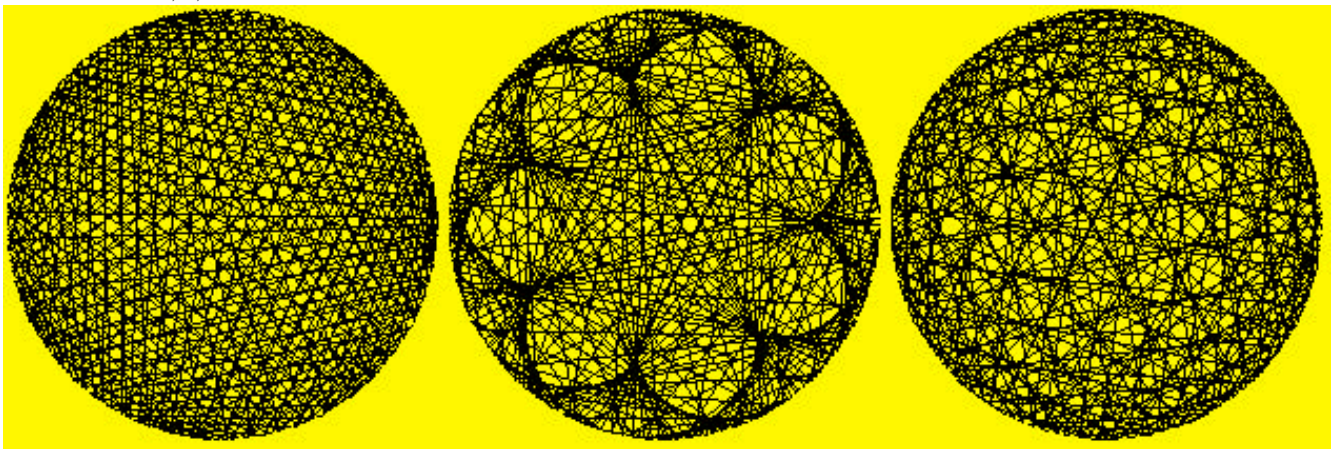


reflections 5, 6, 7.





reflections 8,9,10.



Other reflections, this is where the Principle #1 no longer apply, we lost the b-1 cusps.

References : I have NO references for that, I never found anything published about that.

Program (in BASIC).

This program in FutureBasic will draw one image (with a delay) one after the other. The functions USR 8 and USR 9 are special to that compiler they are the SIN and COS in integers, so for your platform you may replace it by the real sin and real cos. It will be much slower but at least it will work. In this I chose  $p=257$ .

**DEFDBL INT A-Z**

**REM Program to draw reflections of light rays in a cup of coffee**

**REM This is completely obsolete with modern programs like Maple**

**REM or Mathematica of course!**

**REM Still this program on a mac draws them VERY fast.**

**WINDOW #3, , (0, 20) - (640, 480), 3**

**DIM V(2, 300)**

**DIM PAS(300)**

**PLUS=64**

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REM PLUS = 64 is to make the figure being rotated 90 degrees.
REM the coordinates on a screen are not the cartesian plane.
P=257
FOR I=1 TO P
  V(1, I)=0
  V(2, I)=I
  PAS(I)=I
NEXT I
CRB=0
GOSUB "dessine"
"autre courbe"
FOR DR=1 TO P
  V(1, DR)=V(2, DR)
  V(2, DR)=V(2, DR)+PAS(DR)
  IF V(2, DR)>P THEN V(2, DR)=V(2, DR)-P
NEXT DR
GOSUB "dessine"
GOTO "autre courbe"
STOP
"dessine"
PRINT %(405, 460) "reflection #"; crb
crb=crb+1
FOR R=1 TO P
  LONG COLOR 48000, 6000, 6000
REM the choice of that color is blue in fact.
  X0=. 5*(USR 8(V(1, R)+64))+320
  Y0=. 5*(USR 9(V(1, R)+64))+265
  X1=. 5*(USR 8(V(2, R)+64))+320
  Y1=. 5*(USR 9(V(2, R)+64))+265
  PLOT X0, Y0 TO X1, Y1
NEXT R
DELAY 3000
IF INKEY$="s" THEN STOP
CLS
RETURN
REM the numbers 320 and 265 are for the position being just below
REM the comment window on a mac, so you will be able to capture
REM the screens with Command-Shift-3.

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