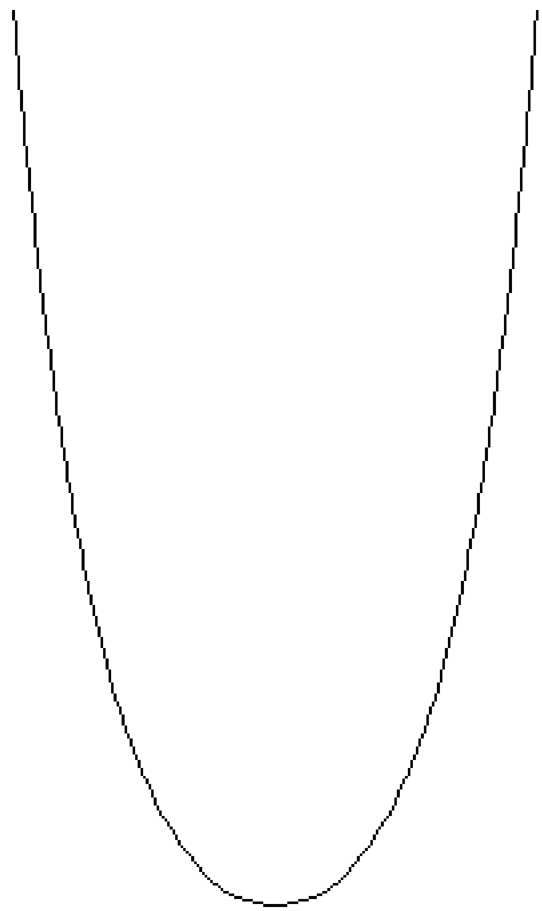
Catenary *

The catenary is also known as the chainette, alysoid, and hyperbolic cosine. It is defined as the graph of the function $y = a\cosh(x/a)$. (Recall that cosh(x) is defined as $(e^x + e^{-x})/2$, where $e = 2.71828\ldots$ is the base of the natural logarithms.



The Catenary

The catenary is the shape an ideal string takes when hanging between two points. By "ideal" is meant that the string is perferctly flexible and inextensible, has no thickness, is of uniform density. In other words the catenary is a mathematical abstraction of the shape of a hanging string, and it closely approximates

*This file is from the 3D-XploreMath project.
Please see http://rsp.math.brandeis.edu/3D-XplorMath/index.html

the shapes of most hanging string-like objects we see, such as ropes, outdoor telecommunication wires, necklaces, chains, etc. For any particular hanging string, we will need to choose the parameter a correctly to model that string.

It is worth to noting that except for scaling there is really only a single catenary. That is, the scaling transformation $(x, y) \mapsto (ax, ay)$ maps the graph of $y = a\cosh(x/a)$ onto the graph of $y = \cosh(x)$. Notice too that the scaling transformation just amounts to a change in the choice of units used to measure distances.

History

Galileo was the first to investigate the catenary, but he mistook it for a parabola. James Bernoulli in 1691 obtained its true form and gave some of its properties. [cf., Robert C. Yates, 1952]

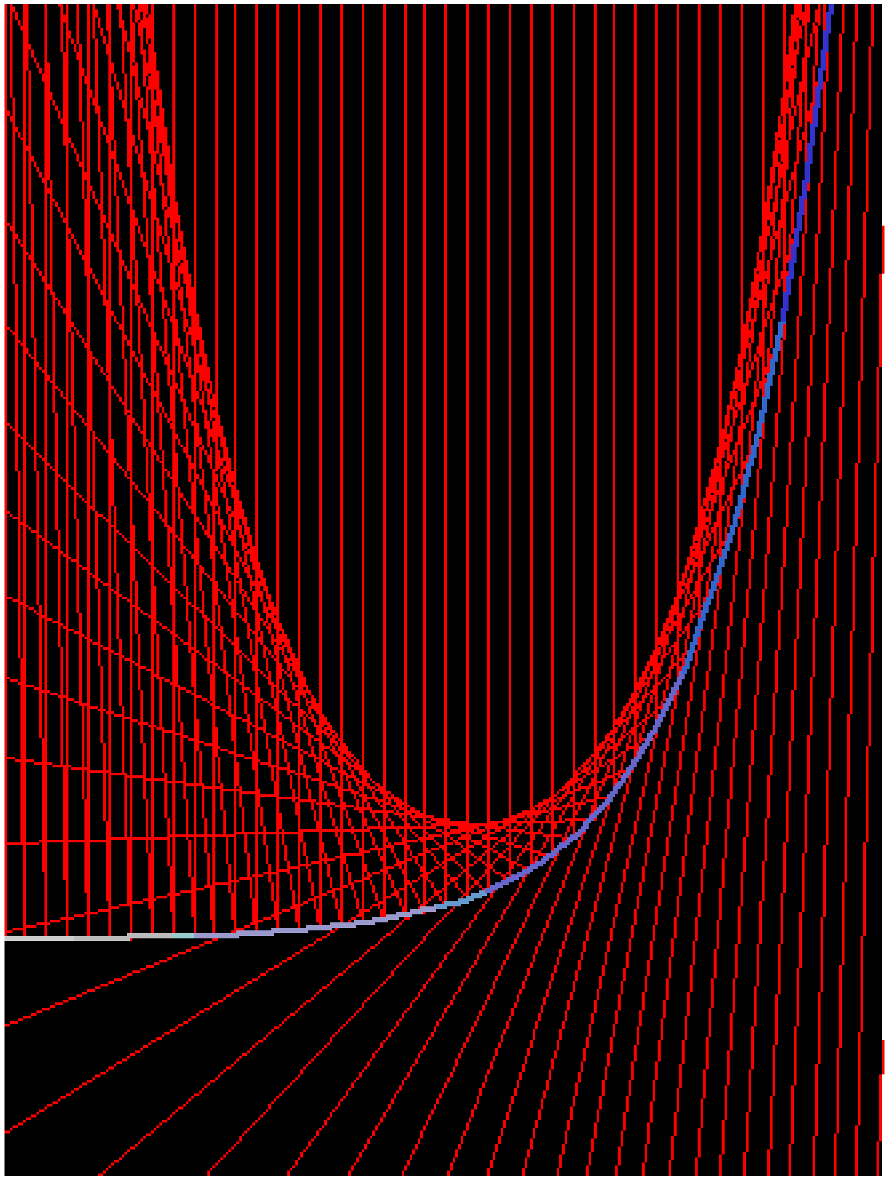
Galileo's suggestion that a heavy rope would hang in the shape of a parabola was disproved by Jungius in 1669, but the true shape of the catenary, was not found until 1690–91, when Huygens, Leibniz and John Bernoulli replied to a challenge by James Bernoulli. David Gregory, the Oxford professor, wrote a comprehensive treatise on the 'catenarian' in 1697. The name was first used by Huygens in a letter to Leibniz in 1690. [cf., E.H.Lockwood, 1961].

[By the way, it is true that if you carefully weight a hanging string so that their is equal weight of string per unit of horizontal distance (rather than per unit of length) then its shape will be a parabola, so Gallileo wasn't so far from the truth.]

The Catenary has numerous interesting properties.

Properties

Caustics

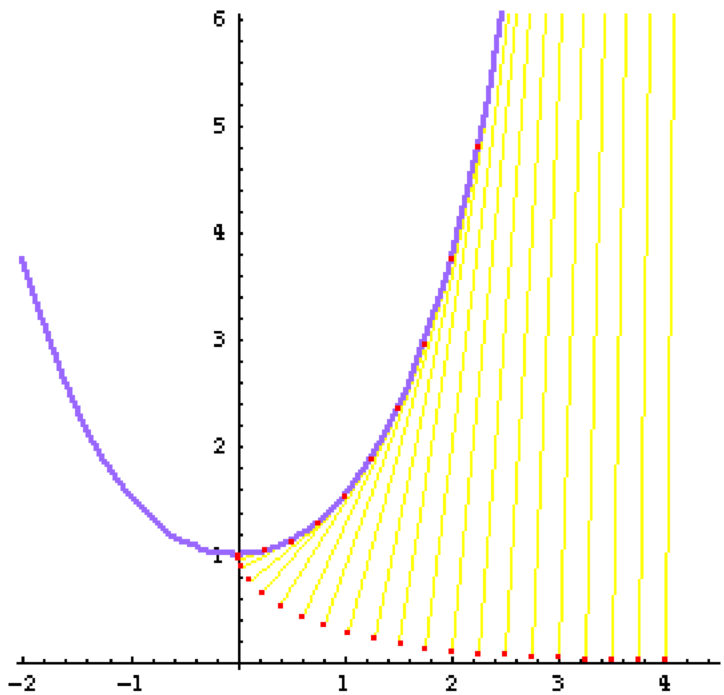


parallel rays above the exponetial curve

The Catacaustic of the exponential curve e^x with light rays from above and parallel to the y axes is the catenary.

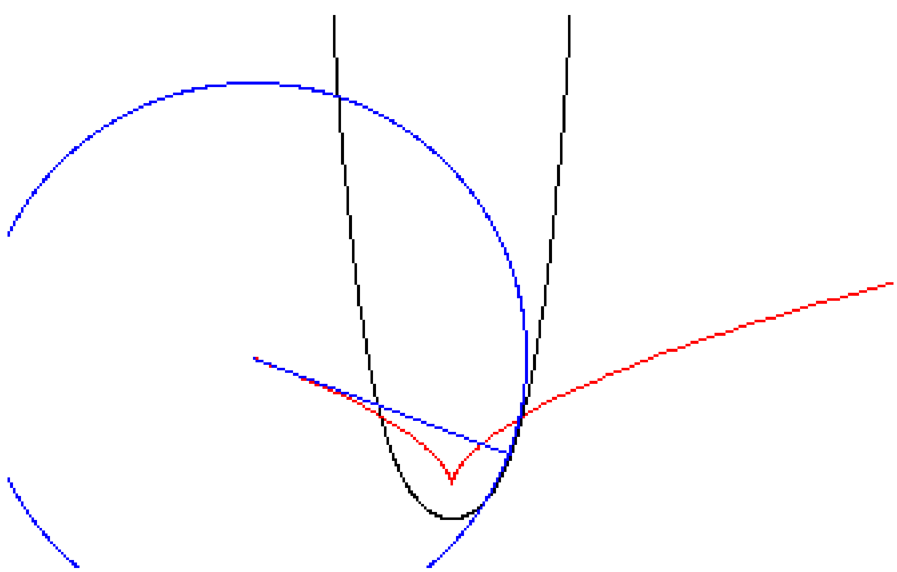
The exponential curve e^x has the interesting property itself. It is the only function whose derivative is itself.

Involute



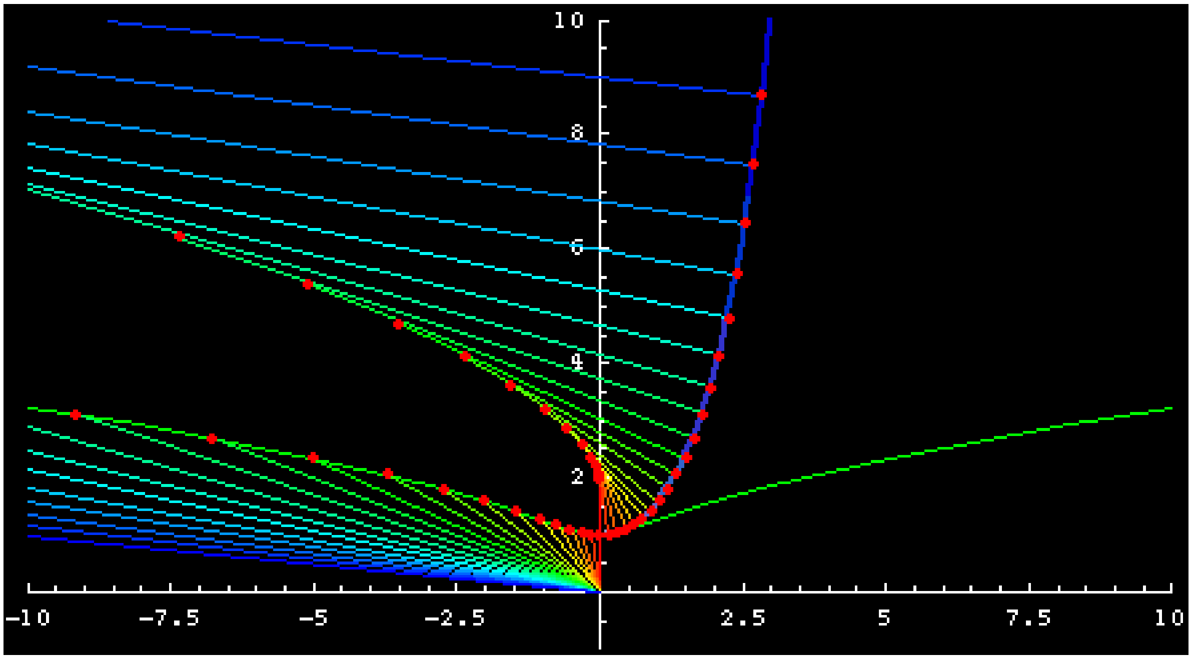
The involute of catenary starting at the vertex is the curve tractrix. (In 3DXM, the involutes of a curve can be shown in the menu Action → Show Involutes.) Note that all involutes are parallel curves of each other. This is a theorem.

Evolute



The evolute of catenary is also the tractrix. (In 3DXM, this can be seen from the menu Action → Show Osculating Circles.)

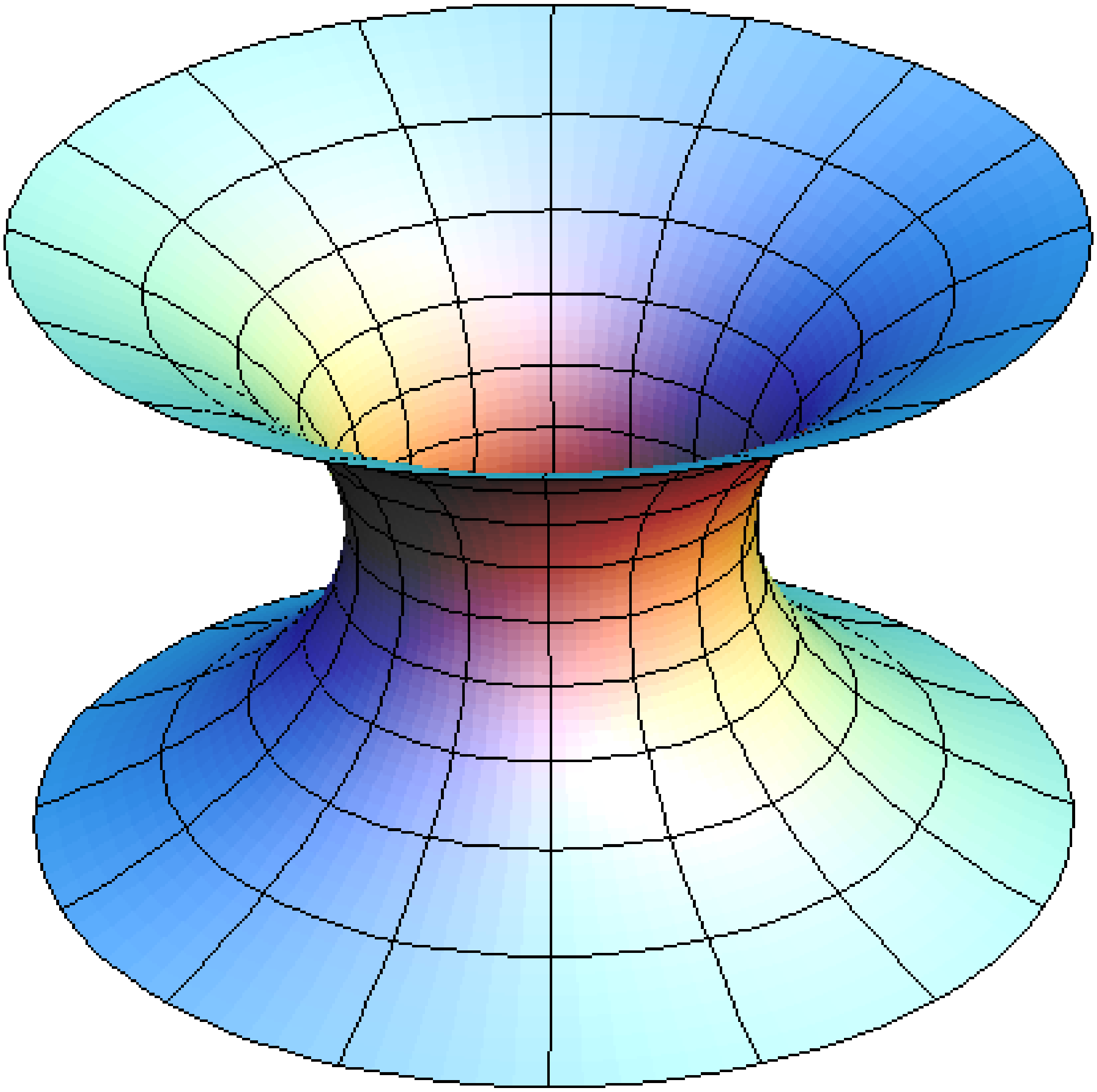
Radial and Kampyle of Eudoxus



The radial of catenary is the Kampyle of Eudoxus. In the figure above, the blue curve is half of catenary. The green curve is Kampyle of Eudoxus. The rainbow lines are the radiuses of osculating circles.

the Kampyle of Eudoxus is defined as the parametric curve $x = -\cosh(t)\sinh(t), y = \cosh(t)$.

Catenoid



If you rotate the graph of $x = \cosh(y)$ about the y-axis, the resulting surface of revolution is a minimal surface, called the Catenoid. It is one endpoint of an interesting morph you can see in 3DXM, by switching to the Surface category, choosing Helicoid-Catenoid from the Surface menu, and then choosing Morph from the Animate menu. If you look closely you will see that during this morph distances and angles on the surface are preserved. See About This Object... in the Documentation menu when Helicoid-Catenoid is selected for a discussion of this.

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