Cissoid *

History

Diocles (250 – ~100 BC) invented this curve to solve the doubling of the cube problem (also know as the the Delian problem). The name cissoid (ivy-shaped) derives from the shape of the curve. Later the method used to generate this curve was generalized, and we call all curves generated in a similar way cissoids.

From Thomas L. Heath's Euclid's Elements translation (1925) (comments on definition 2, book one):

> This curve is assumed to be the same as that by means of which, according to Eutocius, Diocles in his book On burning-glasses solved the problem of doubling the cube.

From Robert C. Yates' Curves and their properties (1952):

> As early as 1689, J. C. Sturm, in his Mathesis Enucleata, gave a mechanical device for the constructions of the cissoid of Diocles.

From E.H.Lockwood A book of Curves (1961):

> The name cissoid ("Ivy-shaped") is mentioned by Geminus in the first century B.C., that is, about a century

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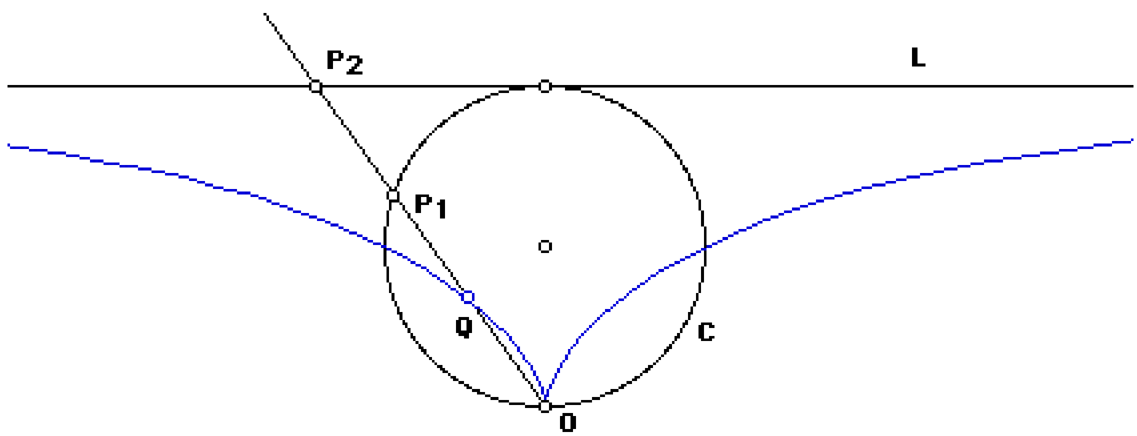
after the death of the inventor Diocles. In the commentaries on the work by Archimedes On the Sphere and the Cylinder, the curve is referred to as Diocles' contribution to the classic problem of doubling the cube. ... Fermat and Roberval constructed the tangent (1634); Huygens and Wallis found the area (1658); while Newton gives it as an example, in his Arithmetica Universalis, of the ancients' attempts at solving cubic problems and again as a specimen in his Enumeratio Linearum Tertii Ordinis.

1 Description



The Cissoid of Diocles is a special case of the general cissoid. It is a cissoid of a circle and a line tangent to the circle with respect to a point on the circle opposite to the tangent point. Here is a step-by-step description of the construction:

1. Let there be given a circle C and a line L tangent to this circle.

2. Let O be the point on the circle opposite to the tangent point.

3. Let P1 be a point on the circle C.

4. Let P2 be the intersection of line [O,P1] and L.

5. Let Q be a point on line[O,P1] such that
$$\text{dist}[\text{O}, \text{Q}] = \text{dist}[\text{P1}, \text{P2}].$$

6. The locus of Q (as P1 moves on C) is the cissoid of Diocles.

An important property to note is that Q and P1 are symmetric with respect to the midpoint of segment [O,P2]. Call this midpoint M. We can reflect every element in the construction around M, which will help us visually see other properties.

2 Formula derivation

Let the given circle C be centered at $(1/2, 0)$ with radius $1/2$. Let the given line L be $x = 1$, and let the given point O be the origin. Let $P1$ be a variable point on the circle, and Q the tracing point on line $[O, P1]$. Let the point $(1, 0)$ be A. We want to describe distance $r = \text{dist}[\text{O}, \text{Q}]$ in terms of the angle $\theta = [A, O, P1]$. This will give us an equation for the Cissoid in polar coordinates (r, θ). From elementary geometry, the triangle $[A, O, P1]$ is a right triangle, so by trignometry, the length of $[O, P1]$ is $\cos(\theta)$. Similarly, triangle $[O, A, P2]$ is a right triangle and the length of $[O, P2]$ is $\frac{1}{\cos(\theta)}$. Since $\text{dist}[\text{O}, \text{Q}] = \text{dist}[\text{O}, \text{P2}] - \text{dist}[\text{O}, \text{P1}]$, we have $\text{dist}[\text{O}, \text{Q}] = \frac{1}{\cos(\theta)} - \cos(\theta)$. Thus the polar equation is $r = \frac{1}{\cos(\theta)} - \cos(\theta)$. If we combine the fractions and use the identity $\sin^2 + \cos^2 = 1$, we arive at an equivalent form: $r = \sin(\theta)\tan(\theta)]$.

3 Formulas

In the following, the cusp is at the origin, and the asymptote is $x = 1$. (So the diameter of the circle is 1.)

Parametric: $(\sin^2(t), \sin^2(t)\tan(t)) \quad -\pi/2 < t < \pi/2.$

Parametric: $\left(\frac{t^2}{(1+t^2)}, \frac{t^3}{(1+t^2)}\right) \quad -\infty < t < \infty$

Polar: $r = \frac{1}{\cos(\theta)} - \cos(\theta) \quad -\pi/2 < t < \pi/2.$

Cartesian equation: $y^2(1-x) = x^3$

The Cissoid has numerous interesting properties.

4 Properties

4.1 Doubling the Cube

Given a segment $[C, B]$, with the help of Cissoid of Diocles we can construct a segment $[C, M]$ such that $\mathrm{dist[C,M]}^3 = 2 * \mathrm{dist[C,B]}^3$. This solves the famous doubling the cube problem.

Step-by-step description:

1. Given two points C and B.

2. Construct a circle $c1$, centered on C and passing through B.

3. Construct points O and A on the circle such that line[O,A] is perpendicular to line $[C, B]$

4. Construct a cissoid of Diocles using circle $c1$, tangent at A, and pole at O.

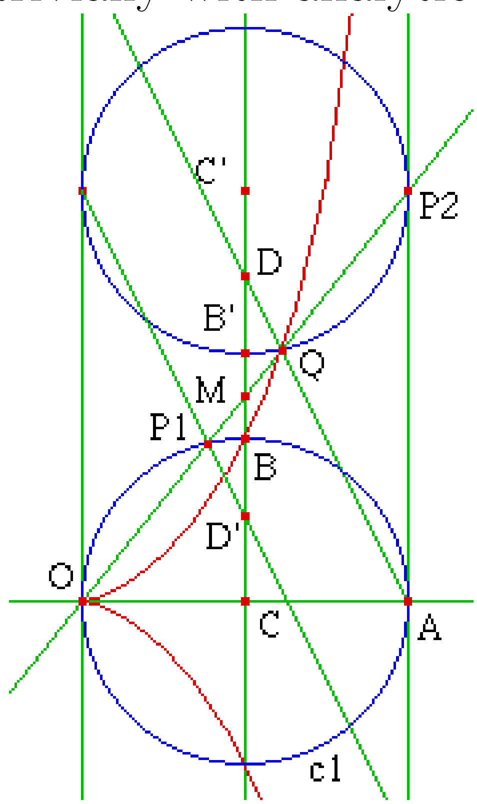
5. Construct point D such that B is the midpoint of segment[C,D].

6. Construct line[A,D]. Let the intersection of cissoid and line[A,D] be Q. (the intersection cannot be found with Greek Ruler and Compass. We assume it is a given.)

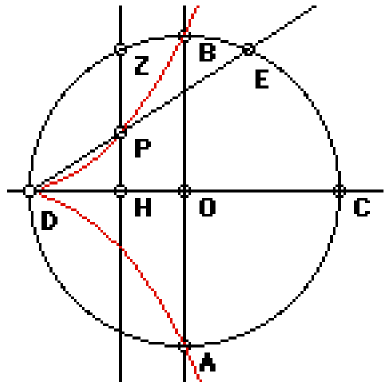
7. Let the intersection of line[C,D] and line[O,Q] be M.

8. $\mathrm{dist}[\mathrm{C}, \mathrm{M}]^3 = 2 * \mathrm{dist}[\mathrm{C}, \mathrm{D}]^3$

This can be proved trivially with analytic geometry.



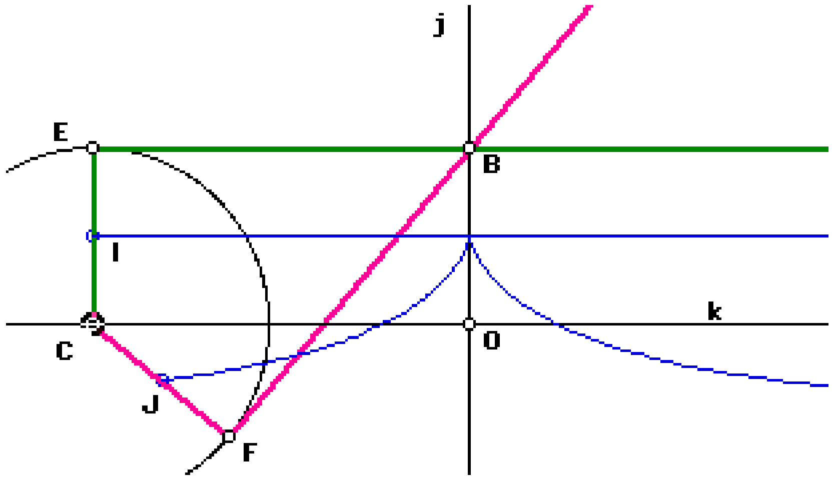
4.2 Diocles' Construction



By some modern common accounts (Morris Kline, Thomas L. Heath), here's how Diocles constructed the curve in his book On Burning-glasses: Let AB and CD be perpendicular diameters of a circle. Let E be a point on arc[B,C], and Z be a point on arc[B,D], such that BE, BZ are equal. Draw ZH perpendicular to CD. Draw ED. Let P be intersection[ZH,ED]. The cissoid is the locus of all points P determined by all positions of E on arc[B,C] and Z on arc[B,D] with arc[B,E]=arc[B,Z]. (the portion of the curve that lies outside of the circle is a latter generalization).

In the curve, we have CH/HZ=HZ/HD=HD/HP. Thus HZ and HD are two mean proportionals between CH and HP. Proof: taking CH/HZ=HZ/HD, we have $CH * HD = HZ^2$. triangle[D,C,Z] is a right triangle since it's a triangle on a circle with one side being the diameter (elementary geometry). We know an angle[D,C,Z] and one side distance[D,C], thus by trignometry of right angles, we can derive all lengths DZ, CZ, and HZ. Substituting the results of computation in $CH * HD = HZ^2$ results an identity. Similarly, we know length HP and find HZ/HD=HD/HP to be an identity.

4.3 Newton's Carpenter's Square and Tangent



Newton showed that cissoid of Diocles can be generated by sliding a right triangle. This method also easily proves the tangent construction.

Step-by-step description:

1. Let there be two distinct fixed points B and O, both on a given line j. (distance[B,O] will be the radius of the cissoid of Diocle we are about to construct.)

2. Let there be a line k passing O and perpendicular to j.

3. Let there be a circle centered on an arbitrary point C on k, with radius OB.

4. There are two tangents of this circle passing B, let the tangent points be E and F.

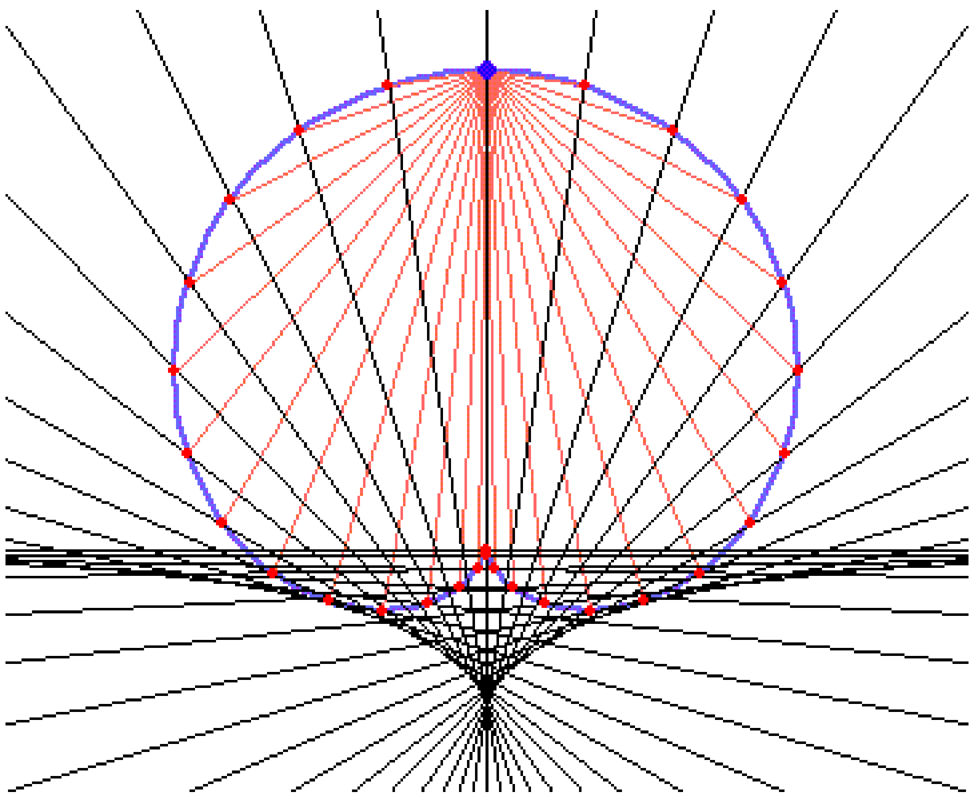
5. Let I be the midpoint between E and the center of the circle. Similarly, let J be the midpoint between F and the center of the circle.

6. The locus of I and J (as C moves on k) is the cissoid of Diocles

and a line. Also, the locus of E and F is the right strophoid.

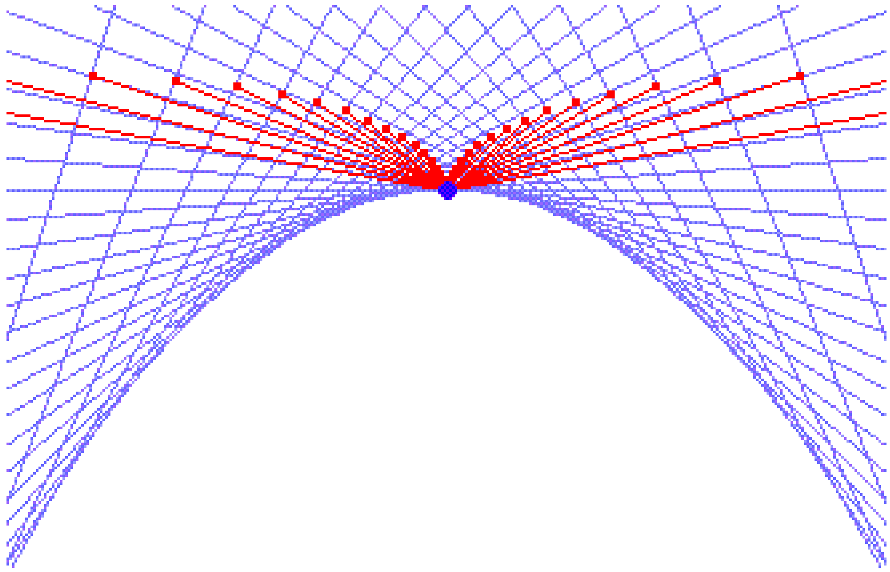
Tangent construction: Think of triangle[C,F,B] as a rigid moving body. The point C moves in the direction of vector[O,C], and point B moves in the direction of vector[B,F]. The intersection H (not shown) of normals of line[O,C] and line[B,F] is its center of rotation. Since the tracing point J is a point on the triangle, thus HJ is normal to the curve.

4.4 Pedal and Cardioid



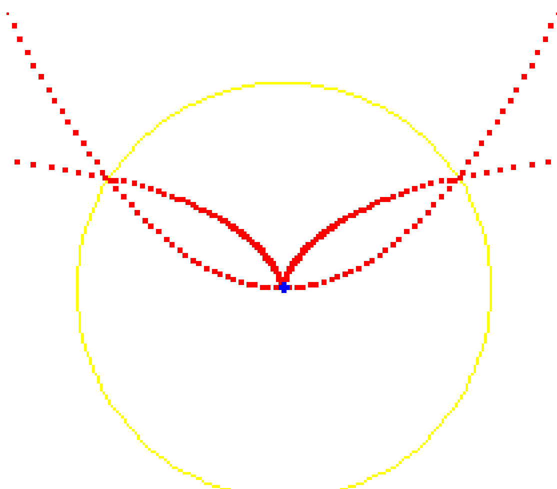
The pedal of a cissoid of Diocles with respect to a point P is the cardioid. If the cissoid's asymptote is the line y = 1 and its cusp is at the origin, then P is at {0,4}. It follows by definition, the negative pedal of a cardioid with respect to a point opposite its cusp is the cissoid of Diocles.

4.5 Negative Pedal and Parabola



The pedal of a parabola with respect to its vertex is the cissoid of Diocles. (and then by definition, the negative pedal of a cissoid of Diocles with respect to its cusp is a parabola.)

4.6 Inversion and Parabola



The inversion of a cissoid of Diocles at cusp is a parabola.

4.7 Roulette of a Parabola

Let there be a fixed parabola. Let there be an equal parabola that rolls on the given parabola in such way that the two parabolas are symmetric to the line of tangency. The vertex of the rolling parabola traces a cissoid of Diocles.

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