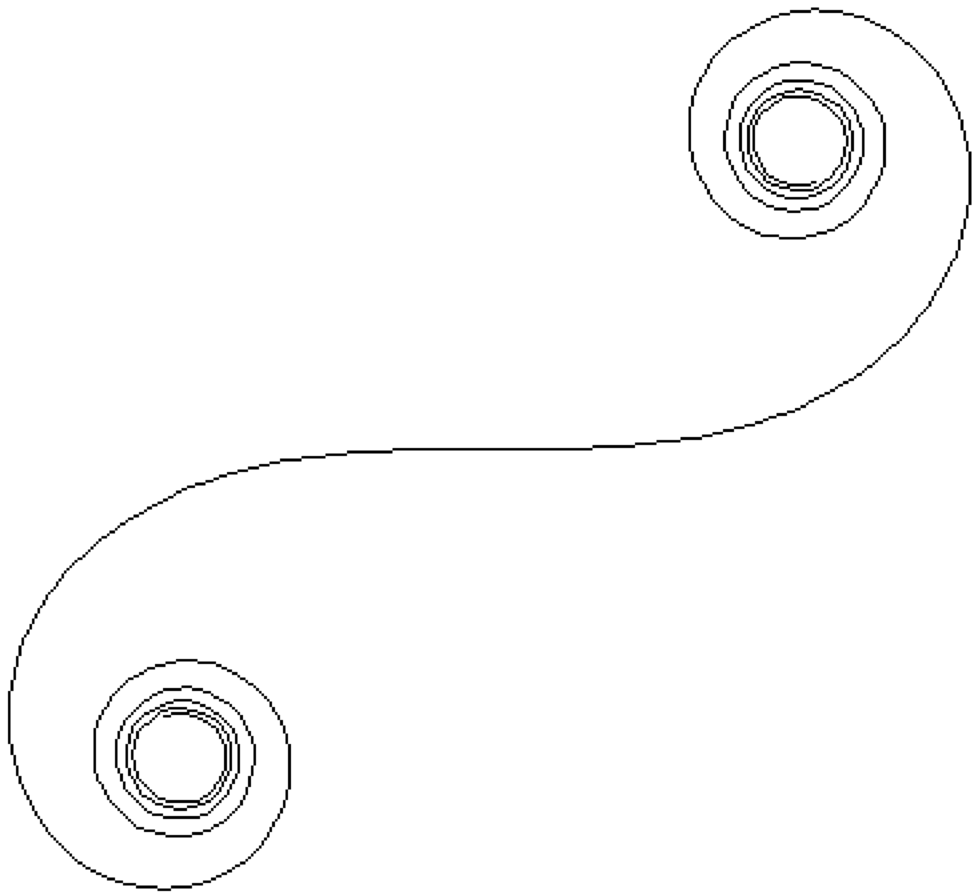
Clothoid *



The Clothoid is also called Euler spiral and Spiral of Cornu, is a curve whose curvature is equal to its arclength. It has the parametric formula:

$$\left(\int_0^t \cos(x^2/2)\,dx, \int_0^t \sin(x^2/2)\,dx\right).$$

*This file is from the 3D-XploreMath project.
Please see http://rsp.math.brandeis.edu/3D-XplorMath/index.html

Discussion

If a plane curve is given by a parametric formula $(f(t), g(t))$, then the length of the part corresponding to a parameter interval $[a, t]$ is $s(t) = \int_a^t \sqrt{f'(\tau)^2 + g'(\tau)^2}\, d\tau$. If we apply this formula to the Clothoid we see that the arclength corresponding to the interval $[0, t]$ is $s(t) = \int_0^t 1\, d\tau = t$, so that the parameter t is precisely the (signed!) arclength measured along the curve from its midpoint, $(0, 0)$.

Next, recall that the curvature κ of a plane curve is defined as the rate of change (with respect to arclength) of the angle θ that its tangent makes with some fixed line (which we can take to be the x-axis). And since the slope $\frac{dy}{dx}$ of the curve is $\tan(\theta)$, and by the chain rule $\frac{dy}{dx} = (dy/dt)/(dx/dt) = \frac{g'}{f'}$, we see that $\theta(t) = \arctan(g'(t)/f'(t))$. So if we assume that parameter t is arclength, then using the formulas for the derivative of the arctangent and of a quotient, we see that:

$$\kappa(t) = \theta'(t) = -g'(t)f''(t) + f'(t)g''(t),$$

(where we have ignored the denominator, since parameterization by arclength implies that it equals unity). Applying this to the Clothoid, we obtain $\kappa(t) = t$. Since the arclength function is also t, this shows that the Clothoid is indeed a curve whose curvature function is equal to its arclength function.

The Fundamental Theorem of Plane Curves

Next let's look at this question from the other direction, and also more generally. Suppose we are given a function $\kappa(t)$. Can we find a plane curve parameterized by arclength $(f(t), g(t))$ such that κ is its curvature function? Recall from above that $\frac{d\theta}{dt} = \kappa$, and of course $\frac{dx}{dt} = f'(t)$ and $\frac{dy}{dt} = g'(t)$. Now, since $(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 1$, while $\frac{dy}{dt}/\frac{dx}{dt} = dy/dx = \tan(\theta)$, it follows from elementary trigonometry that $\frac{dx}{dt} = \cos(\theta)$ while $\frac{dy}{dt} = \sin(\theta)$. Thus we have the following system of three differential equations for the three functions $\theta(t)$, $f(t)$, and $g(t)$:

$$\begin{aligned}
\theta'(t) &= \kappa(t)\\
f'(t) &= \cos(\theta(t))\\
g'(t) &= \sin(\theta(t)).
\end{aligned}$$

The first equation has the general solution $\theta(\tau) = \theta_0 + \int_0^\tau \kappa(\sigma)\,d\sigma$, and substituting this in the other two equations, we find that the general solutions for f and g are given by:

$$\begin{aligned}
f(t) &= x_0 + \int_0^t \cos(\theta_0 + \int_0^\tau \kappa(\sigma)\,d\sigma)\,d\tau\\
g(t) &= y_0 + \int_0^t \sin(\theta_0 + \int_0^\tau \kappa(\sigma)\,d\sigma)\,d\tau.
\end{aligned}$$

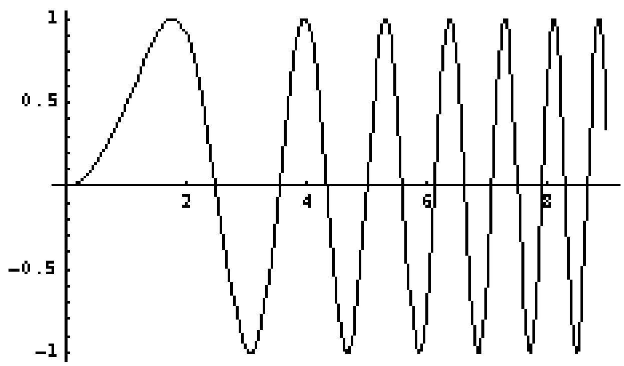
This is an elegant explicit solution to our question! It shows that not only is there a solution to our question (say the one obtained by setting x_0, y_0 and θ_0 all equal to zero), but also that the solution is unique up to a translation (by (x_0, y_0)) and a rotation (by θ_0), that is unique up to a general rigid motion.

This fact has a name—it is called The Fundamental Theorem of Plane Curves. It tell us us that most geometric and most economical descriptions of plane curves is not via parametric equations, which have a lot of redundancy, but rather by the single function κ that gives the curvature as a function of arclength.

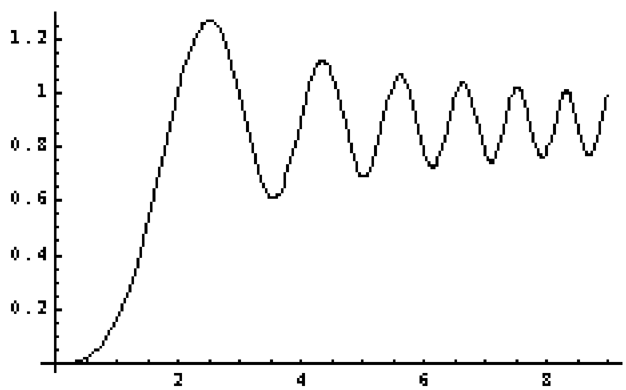
Exercise Take $\kappa(t) = t$ and check that the above formulas give the parametric equations for the Clothoid in this case.

Back to the Clothoid

We close with a few more details about the Clothoid. First, here is a plot of the integrand $\sin(x^2/2)$:



and next a plot of its indefinite integral, $\int_0^t \sin(x^2/2)\,dx$, the so-called Fresnel integral:



From this plot we see that the y-coordinate oscillates. Its limit as t goes to infinity is $\sqrt{\pi}/2$, from which we see that the centers of the two spirals of the Clothoid are at $(\pm\sqrt{\pi}/2, \pm\sqrt{\pi}/2)$.

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