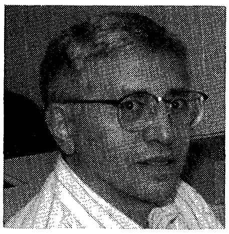
Spirals and Conchospirals in the Flight of Insects

Khristo N. Boyadzhiev



Khristo Boyadzhiev (k-boyadzhiev@onu.edu) received his MS in mathematics and his Ph.D. in functional analysis under the direction of Yaroslav Tagamlitski at the university of Sofia, Bulgaria (1978). His main interests are in operator theory and classical analysis, but he also enjoys art, history and the Internet. A member of the mathematics department at Ohio Northern University since 1990, he coordinates the calculus program and the department seminar.

"Spiral forms are encountered so frequently that it's impossible even to name the variety of their manifestations. . . . our world is spiral. However, we have a long way to go in explaining the mystery of how spirals emerge and persist."

N. Bourbaki [2]

1. Introduction. Everybody has seen how moths spiral around a night light. If they are simply attracted to the light, they would fly on a straight line. The spiral trajectory is explained by the way some primitive animals orient themselves by a point light source [4]. Insects have compound eyes consisting of many ommatidia, which are single detectors of light. Beams of light stimulate a small group of ommatidia, orthogonal to the beams, and this determines the angular position relative to the light source. When the beams are parallel and the insect wants to go in straight line, it needs to fly in such a way that the same group of ommatidia will stay activated. That is, it needs to keep a constant angle with the beams, see Figure 1.

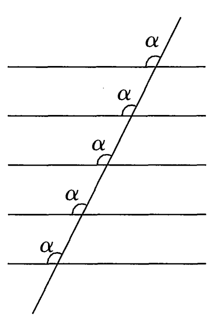


Figure 1

We will use α for the angle between the direction of flight and the direction towards the light source. Naturally, $0 \leq \alpha \leq \pi$. For millions of years insects used the moon and the sun for orientation. Their beams are practically parallel and help maintain a straight path. The situation changed when humans introduced other light sources. If a moth keeps a constant angle α to the radial beams of an electric light, then it flies on a straight line only when $\alpha = 0$, flying directly towards the light, or $\alpha = \pi$, flying directly away from the light. It will approach the light on a spiral when $0 < \alpha < \pi/2$ (Figure 2) or fly away on a spiral when $\pi/2 < \alpha < \pi$. In the special case when $\alpha = \pi/2$, it flies in circles around the light. When the motion is in a plane, the trajectory is an equiangular spiral, as confirmed by observations. We present here a study of the general case, orientation by a light source in 3-space. First, though, we will consider the planar motion.

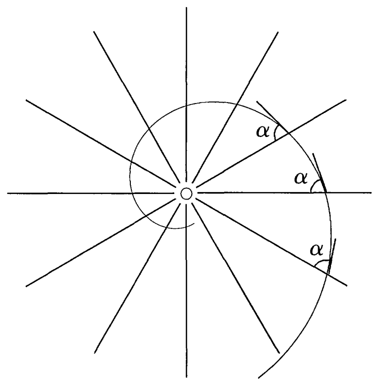


Figure 2

2. Planar law of motion. We assume that the light source is at the origin of the coordinate system. Suppose our insect with radius vector $\bar{x} = (x, y)$ flies by the law

$$x(t) = r(t)\cos(\theta(t)), y(t) = r(t)\sin(\theta(t)) \tag{2.1}$$

where t is time, r, θ are the polar coordinates, and we allow θ to run from $-\infty$ to $+\infty$. The direction of flight is given by the tangent vector $\bar{x}'$ and we want the angle α between $\bar{x}'$ and $-\bar{x}$ (the direction toward the light) to stay constant (Figure 3). This way we have

$$x' = r'\cos(\theta) - r\sin(\theta)\theta', y' = r'\sin(\theta) + r\cos(\theta)\theta'$$

$$\frac{xx' + yy'}{\sqrt{x^2+y^2}\sqrt{(x')^2+(y')^2}} = \frac{r'}{\sqrt{(r')^2+(\theta')^2 r^2}} = -\cos\alpha, \tag{2.2}$$

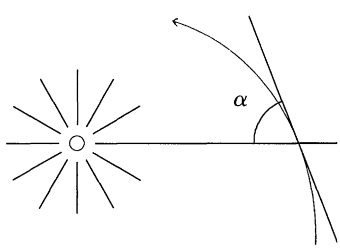


Figure 3

which is the general differential equation of equiangular motion. A natural assumption is that the insect flies with a constant speed, i.e.

$$\|\bar{x}'\|^2 = (x')^2 + (y')^2 = (r')^2 + (\theta')^2 r^2 = v^2 = \text{constant}. \tag{2.3}$$

It is easy now to solve equation (2.2). It turns into

$$r' = -v \cos \alpha$$

and therefore, with $r_0 = r(0)$,

$$r(t) = -(v \cos \alpha) t + r_0. \tag{2.4}$$

Next, to find $\theta(t)$ we substitute $r' = -v \cos \alpha$ in (2.3):

$$v^2 \cos^2 \alpha + (\theta')^2 r^2 = v^2, (\theta')^2 r^2 = v^2 \sin^2 \alpha, \theta' r = v \sin \alpha,$$

$$\theta(t) = \int \frac{v \sin \alpha}{r(t)} dt = \int \frac{v \sin \alpha}{r_0 - (v \cos \alpha) t} dt = -\tan \alpha \ln(r_0 - vt \cos \alpha) + C.$$

Here we need $r_0 > vt \cos \alpha$, i.e. $0 \le t < r_0 / v \cos \alpha$, which is the natural time interval of the flight. Setting $\theta_0 = \theta(0)$ we find

$$\theta(t) = \theta_0 - \ln\left(1 - \frac{vt}{r_0} \cos \alpha\right) \tan \alpha. \tag{2.5}$$

When we substitute (2.4) and (2.5) in (2.1) we find the law of motion in terms of t:

$$x(t) = (r_0 - vt \cos \alpha) \cos\left(\theta_0 - \ln\left(1 - \frac{vt}{r_0} \cos \alpha\right) \tan \alpha\right)$$

$$y(t) = (r_0 - vt \cos \alpha) \sin\left(\theta_0 - \ln\left(1 - \frac{vt}{r_0} \cos \alpha\right) \tan \alpha\right).$$

How realistic is this model? It works fairly well when the insect is small (small inertia) and its velocity is small. For a larger, faster insect, such as a bumble bee, there will be distortions. Note also that the real time of flight is "short," the flight will end at $t = r_0 / v \cos \alpha$, although (2.5) requires infinitely many turns around the light source, as

$$-\ln\left(1 - \frac{vt}{r_0} \cos \alpha\right) \to +\infty \text{ when } t \to r_0 / v \cos \alpha.$$

The trajectory of the flight in polar coordinates is an equation of the form

$$r = r(\theta), -\infty < \theta < \infty.$$

To find it, we eliminate t from (2.4) and (2.5). This gives

$$r(\theta) = r_0 \, e^{\cot \alpha (\theta_0 - \theta)}, \tag{2.6}$$

which shows that the trajectory is an equiangular spiral. If we need only the trajectory, we can get (2.6) directly from (2.2) by choosing θ for a parameter and setting $\theta' = 1$. Then from (2.2)

$$\frac{(r')^2 + r^2}{(r')^2} = \frac{1}{\cos^2 \alpha}, (r/r')^2 = \tan^2 \alpha, \frac{r'}{r} = -\cot \alpha,$$

where $r' = dr/dt = dr/d\theta$. Integrating we come to (2.6). Two examples are shown on Figures 4 and 5.

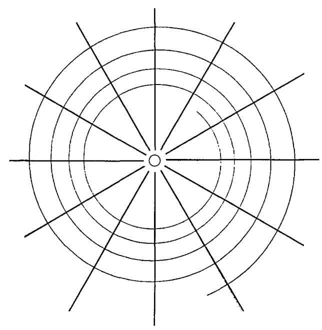


Figure 4. $\alpha = 82°$.

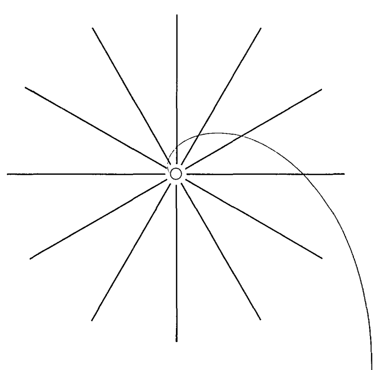


Figure 5. $\alpha = 45°$.

3. Flight in 3-dimensions. The radius vector of the moth is now $\bar{x} = (x, y, z)$. In cylindrical coordinates

$$x = r\cos\theta, y = r\sin\theta, z = z.$$

Assume again that the vector $-\bar{x}$ and the tangent vector $\bar{x}'$ have a constant angle α:

$$\frac{\bar{x}\cdot\bar{x}'}{\|\bar{x}\|\,\|\bar{x}'\|} = \frac{rr' + zz'}{\sqrt{r^2 + z^2}\sqrt{(r')^2 + (\theta')^2 r^2 + (z')^2}} = -\cos\alpha. \tag{3.1}$$

This time we want to find the trajectory first. For this purpose we consider θ as an independent variable. Set $\theta' = d\theta/dt = 1$ and assume that all derivatives are taken with respect to θ. The equation, however, is still very general. For instance, if $\alpha = \pi/2$, then it becomes $rr' + zz' = 0$ and every smooth curve on the sphere $r^2 + z^2 = c$, c a constant, is a solution. Obviously, further restrictions are needed. We turn again to our example, the flying moth, and study it more closely. Orienting itself by the light source, it tries to keep only one group of ommatidia activated, so the beam hits the same spot on the eye all the time. Assuming that the light source is above the insect when it starts to fly ($z < 0$), we come to a natural hypothesis, confirmed by observation.

3D Hypothesis. In its flight the insect keeps a constant angle, say β, between the direction to the light source and the direction "down-up."

This way it does not have to turn its head during flight when changing altitude, see Figure 6. More precisely, β is the angle between the negative radius vector $-\bar{x}$ (the direction to the light) and the positive z-axis. In our setting, $0 \leq \beta \leq \pi/2$. The unit vector on the positive vertical axis is $\bar{k} = (0, 0, 1)$. Therefore, since $z < 0$,

$$\frac{\bar{k} \cdot \bar{x}}{\|\bar{k}\| \, \|\bar{x}\|} = \frac{z}{\sqrt{r^2 + z^2}} = -\cos\beta, \qquad \frac{z^2 + r^2}{z^2} = \frac{1}{\cos^2 \beta}$$

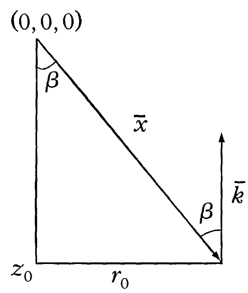


Figure 6

which gives

$$r^2 = z^2 \tan^2 \beta, \quad or \quad x^2 + y^2 = (\tan^2 \beta) z^2 \tag{3.2}$$

and we see that the trajectory lies on the right circular cone with vertex $(0, 0, 0)$ and opening of 2β. From here we also find that

$$r^2 + z^2 = z^2/\cos^2 \beta \text{ and } rr' + zz' = (r^2 + z^2)'/2 = zz'/\cos^2 \beta.$$

Substituting in (3.1) we come to the remarkable equation

$$\frac{z'}{\sqrt{(r')^2 + r^2 + (z')^2}} = \cos\alpha \cos\beta \tag{3.3}$$

which can be written as

$$\frac{\bar{k} \cdot \bar{x}'}{\|\bar{k}\| \, \|\bar{x}'\|} = \cos\gamma$$

with $\cos\gamma = \cos\alpha \cos\beta$. It shows that the tangent vector $\bar{x}' = (x', y', z')$ keeps a constant angle γ with the vector $\bar{k}$, the "down-up" direction. Further, (3.2) gives

$$r = -z\tan\beta,\ r' = -z'\tan\beta,\ (r')^2 + (z')^2 = (z')^2/\cos^2\beta$$

(note that $z < 0$, $\sqrt{z^2} = -z$) and therefore

$$(r')^2 + r^2 + (z')^2 = \left((z')^2 + z^2\sin^2\beta\right)/\cos^2\beta.$$

From (3.3) we obtain

$$\frac{z'}{\sqrt{(z')^2 + z^2\sin^2\beta}} = +\cos\alpha, \qquad \frac{z^2\sin^2\beta + (z')^2}{(z')^2} = \frac{1}{\cos^2\alpha},$$

$$(z/z')^2\sin^2\beta = \tan^2\alpha,$$

$$\frac{z'}{z} = -\frac{\sin\beta}{\tan\alpha}, \qquad \frac{dz}{z} = -\sin\beta\cot\alpha\, d\theta, \qquad \ln|z| = -\sin\beta\cot\alpha\theta + C$$

or

$$z = z_0\, e^{-(\sin\beta\cot\alpha)\theta}.$$

Then

$$r = -z\tan\beta = -\tan\beta z_0\, e^{-(\sin\beta\cot\alpha)\theta}$$

and setting $r_0 = -\tan\beta z_0$, $m = \sin\beta\cot\alpha$, we come to the **equations of the trajectory**

$$x = r_0\, e^{-m\theta}\cos\theta, \qquad y = r_0\, e^{-m\theta}\sin\theta, \qquad z = z_0\, e^{-m\theta}.$$

The curve is a conchospiral winding on the cone mentioned above. It cuts all lines through the vertex at angle α. Figures 7, 8, 9 show three conchospirals for $\beta = \pi/4 = 45°$ and three different values of α. One interesting case is $\alpha = \pi/2$. Then $m = 0$ and we have motion on the circle

$$x = r_0\cos\theta, \qquad y = r_0\sin\theta$$

in the plane $z = z_0$. A fly circling under a light bulb is a typical example.

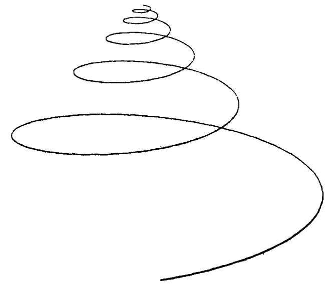


Figure 7. $\alpha = 82°$.

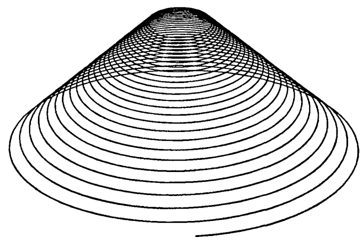


Figure 8. $\alpha = 88°$.

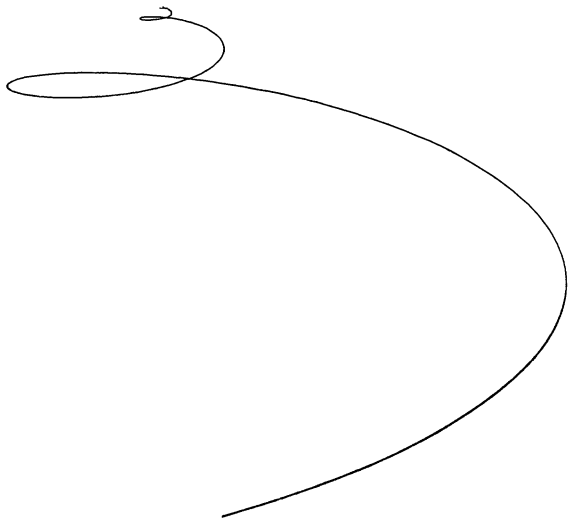


Figure 9. $\alpha = 63°$.

Now we turn to the law of motion and assume as before that the speed, $v = \|\bar{x}'\|$, is constant. From (3.3) we find

$$z' = dz/dt = v\cos\alpha\cos\beta,\ z = (v\cos\alpha\cos\beta)t + z_0, \tag{3.4}$$

$$r = -z\tan\beta = -(v\cos\alpha\sin\beta)t + r_0. \tag{3.5}$$

We also need $\theta = \theta(t)$. By substituting z' and r' in the equation

$$(r')^2 + (\theta')^2 r^2 + (z')^2 = v^2$$

we get

$$v^2\cos^2\alpha + (\theta')^2 r^2 = v^2 \text{ or } \theta' r = v\sin\alpha$$

so that

$$\theta = \int \frac{v\sin\alpha\, dt}{r_0 - (v\cos\alpha\sin\beta)t} = -\frac{\tan\alpha}{\sin\beta}\ln\left(1 + \frac{vt}{z_0}\cos\alpha\cos\beta\right) + \theta_0. \tag{3.6}$$

(We have factored out $r_0 = -z_0\tan\beta$ inside the logarithm.)

Finally, setting for brevity $a = v\cos\alpha\cos\beta$, $b = v\cos\alpha\sin\beta$, $c = \tan\alpha/\sin\beta$ in (3.4), (3.5), and (3.6) we come to the law of motion

$$x = (r_0 - bt)\cos[\theta_0 - c\ln(1 + at/z_0)]$$
$$y = (r_0 - bt)\sin[\theta_0 - c\ln(1 + at/z_0)]$$
$$z = z_0 + at.$$

The case $\alpha = \pi/2$ was mentioned above. When $0 \le \alpha < \pi/2$, then $a > 0$. The time t starts from 0 and approaches the positive limit $-z_0/a$ as the moth approaches, spiraling towards the light source. In the ideal mathematical model it would spiral endlessly, closer and closer to the vertex, as $\ln(1 + at/z_0) \to -\infty$. In real life the moth hits the bulb.

When $\pi/2 < \alpha \le \pi$ we have $a < 0$, $b < 0$. As time grows from 0 to $+\infty$, we have $x, y \to +\infty$ and $z \to -\infty$. The moth flies away from the light!

Here are two extreme cases for the angle β.

$\beta = \pi/2$. Then $z = 0$ and the curve is on the x, y plane as studied in Section 2.

$\beta = 0$. Then $r = 0$ from (3.5) and $x = y = 0$, $z = -(v\cos\alpha)t + z_0$. We have a vertical motion.

4. Comments. When the insect orients itself by the parallel rays of the moon, (3.1) does not apply. We cannot use the moon for the center of our coordinate system and identify its rays with the radius vector. However, we do not need any equations in this case. On a regular trip, the insect would maintain a constant altitude i.e. fly in the plane $z = z_0$. Keeping a constant angle with the moon's rays is equivalent to keeping a constant angle with their parallel projections on $z = z_0$. The flight is on a straight line.

When the insect orients itself by a point light source it can fly at a constant altitude only when $\alpha = \pi/2$ or $z_0 = 0$ (the case $\beta = \pi/2$) as discussed above. If $\alpha \ne \pi/2$ and $z_0 < 0$, then it *has to* change altitude if it wants to keep a constant angle between the direction of flight and the direction to the source. Indeed, suppose $\alpha < \pi/2$ and it stays in the plane $z = z_0$. As the moth spirals closer and closer to the z-axis, $\bar{x}$ and $\bar{x}'$ will become more and more perpendicular, which contradicts (3.1). This contradiction can be formally derived from (3.1) if we assume constant speed and set $z = z_0 < 0$, $z' = 0$. The equation becomes

$$\frac{r\,dr}{\sqrt{r^2 + z_0^2}} = -v\cos\alpha\,dt$$

which integrates to

$$\sqrt{r^2 + z_0^2} = \sqrt{r_0^2 + z_0^2} - (v\cos\alpha)t$$

and as t grows, starting from $t = 0$, the right side decreases to 0, while the left side is always greater than $|z_0| > 0$. When $\alpha > \pi/2$ and $\cos\alpha < 0$, we can reverse the time $t \to -t$ and use the same contradiction by symmetry (i.e. the insect flies backward and cuts an angle $\pi - \alpha$).

If we want to describe a flight that starts above the light source (i.e. $z_0 > 0$) we need to adjust the 3D hypothesis by assuming that $\pi/2 < \beta < \pi$. Some signs will change in the law of motion. We note that the orientation principle discussed here is similar to compass navigation, where light beams are replaced by meridians. If an airplane follows a constant compass direction, its path cuts all meridians at a constant angle. Therefore, the path is a spiral on a sphere, or a rhumb line [7].

Equiangular spirals, also known as logarithmic spirals, appear in many different situations. We list some references here for those interested: [1], [2], [3], [5], [6], [8].

References

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Digits in Triangular Squares

Professor Dipendra Sengupta (Elizabeth City State University, Elizabeth City NC 27909, dsengupta@umfort.cs.ecsu.edu) has computed the first forty-nine integers that are simultaneously triangular and square, as $36 = 6^2 = (8 \cdot 9)/2$ and $1225 = 35^2 = (49 \cdot 50)/2$. Among them is

$$33388478175597782548449 61,$$

twenty-six digits without a zero. The probability of that is only $(.9)^{26} = .0646\ldots$ but we should not be surprised: squares and triangles are very angular and hence incompatible with ovals. On the other hand,

$$1510463834933252340900092196 13956$$

has no sevens, which is inexplicable.