

Linear Algebra Notes

By Xah Lee.

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http://xahlee.org/PageTwo_dir/more.html

1 System of Linear Equations

■ Solutions to a system of linear equations

There are only 3 possibilities regarding the solutions to a system of linear equations:

- No solution.
- Exactly one solution.
- Infinitely many solutions.

■ Definition: Echelon Form, Reduced Echelon form, and Pivot Positions.

A leading entry of a row refers to the leftmost nonzero entry (in a nonzero row).

A rectangular matrix is in **echelon form** if it has the following two properties:

1. All nonzero rows are above any rows of all zeros.
2. If the leading entry of row n is in column m , then the leading entry for row $n+1$ must be in column $m+1$ or greater.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form**:

1. The leading entry in each nonzero row is 1.
2. Each leading 1 is the only nonzero entry in its column.

The leading entries of a matrix in echelon form are called **Pivot Positions**.

■ Existence and Uniqueness Theorem

- Each matrix is row equivalent to one and only one reduced echelon matrix. († proof in Appendix)
- A linear system is consistent iff the rightmost column of the augmented matrix is not a pivot column, that is, iff an echelon form of the augmented matrix has no row of the form $[0 \dots 0 \ b]$ with b nonzero.
- If a linear system is consistent, then the solution set contains either (1) a unique solution, when there are no free variables, or (2) infinitely many solutions, when there is at least one free variable.

by Xah: I like to see a (formal) proof of this.

2 Vectors and Matrix Equations

■ Span

If $\vec{a}_1, \dots, \vec{a}_n$ are vectors in \mathbb{R}^m and x_1, \dots, x_n are scalars, then $\text{Span}[\{\vec{a}_1, \dots, \vec{a}_n\}] := x_1 \vec{a}_1 + \dots + x_n \vec{a}_n$.

$\text{Span}[\{\vec{a}_1, \dots, \vec{a}_n\}]$ is a subset of \mathbb{R}^m .

If $\text{Span}[\{\vec{a}_1, \dots, \vec{a}_n\}] = \mathbb{R}^m$, we say the set of vectors $\{\vec{a}_1, \dots, \vec{a}_n\}$ spans \mathbb{R}^m .

■ Definition of $A\vec{x}$

If A is an $m \times n$ matrix, with columns $\vec{a}_1, \dots, \vec{a}_n$, and \vec{x} is in \mathbb{R}^n , then the vector $A\vec{x}$ is defined to be $\sum_{i=1}^n x_i \vec{a}_i$.

■ Equivalent equations

Given an $m \times n$ matrix A , with columns $\vec{a}_1, \dots, \vec{a}_n$ and given \vec{b} is in \mathbb{R}^m , the following equations all denote the same thing.

- The matrix equation $A\vec{x} = \vec{b}$
- the vector equation $\sum_{i=1}^n x_i \vec{a}_i = \vec{b}$
- the system of linear equations whose augmented matrix is $\{\vec{a}_1 \dots \vec{a}_n \vec{b}\}$

■ Equivalent Statements

Let A be a $m \times n$ matrix. Then the following statements are logically equivalent.

- For each b in \mathbb{R}^m , the equation $A\vec{x} = \vec{b}$ is consistent.
- The columns of A span \mathbb{R}^m .
- A has a pivot position in every row.

By Xah: They are simply logically equivalent statements. Whether they are true is another matter. For (3), think of A as a set of vectors, and a pivot in every row means this set contain all the basis vectors in \mathbb{R}^m , thus it span \mathbb{R}^m . It is not sufficient if A has a pivot position in every column. Example: $\vec{a}_1 := \{1,0,0\}$, $\vec{a}_2 := \{0,1,0\}$ does not span \mathbb{R}^3 .

■ Distribution Property of Vectors

If A is an $m \times n$ matrix, \vec{u} and \vec{v} are vectors in \mathbb{R}^n , and c is a scalar, then

- $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$
- $A(c\vec{u}) = c(A\vec{u})$

■ A Theorem on Homogeneous Equation

A system of linear equations is said to be homogeneous if it can be written in the form $A\vec{x} = \vec{0}$. The homogeneous system $A\vec{x} = \vec{0}$ has a nonzero solution iff the system has at least one free variable.

by Xah: This is so because a consistent system either has one unique solution or infinite number of solutions. For the latter, there must be a free variable.

Suppose the equation $A\vec{x} = \vec{b}$ is consistent for some given \vec{b} , and let \vec{p} be a solution. Then the solution set of $A\vec{x} = \vec{b}$ is the set of all vectors of the form $\vec{p} + \vec{v}_h$ where \vec{v}_h is any solution of the homogeneous equation $A\vec{x} = \vec{0}$.

Proof by Xah: We have

$$A\vec{p} = \vec{b} \text{ and } A\vec{v}_h = \vec{0}$$

$$A\vec{p} + A\vec{v}_h = \vec{b} + \vec{0} = \vec{b}$$

$$A(\vec{p} + \vec{v}_h) = \vec{b}$$

thus $(\vec{p} + \vec{v}_h)$ is a solution to $A\vec{x} = \vec{b}$.

Now we show that there can be no solution other than $(\vec{p} + \vec{v}_h)$. Suppose \vec{c} is a solution to $A\vec{x} = \vec{b}$. Then

$$A\vec{c} = \vec{b}$$

$$A\vec{c} - A\vec{p} = \vec{b} - \vec{b} = \vec{0}$$

$$A(\vec{c} - \vec{p}) = \vec{0}$$

This shows that $(\vec{c} - \vec{p}) = \vec{v}_h$ for some \vec{v}_h

$$(\vec{c} - \vec{p}) = \vec{v}_h$$

$$\vec{c} = \vec{v}_h + \vec{p}$$

End of proof.

■ Definition of Linearly Dependence

A set of vectors $\{\vec{a}_1, \dots, \vec{a}_n\}$ in vector space V is said to be linearly independent if the vector equation

$$\sum_{i=1}^n x_i \vec{a}_i = \vec{0}$$

has only the trivial solution $\vec{x} = \vec{0}$. Otherwise, the set is said to be linearly dependent.

■ Characterization of Linearly Dependent Sets

- A set $S := \{\vec{a}_1, \dots, \vec{a}_n\}$ of two or more vectors is linearly dependent iff at least one of the vectors in S is a linear combination of the others.
- Any set $\{\vec{a}_1, \dots, \vec{a}_n\}$ in \mathbb{R}^m is linearly dependent if $n > m$.
- If a set $S := \{\vec{a}_1, \dots, \vec{a}_n\}$ in \mathbb{R}^m contains the zero vector, then the set is linearly dependent.

Problem: Given a set of vectors with one element a linear combination of others. How to determine which one and its linear combination? In general, given a set of vectors, how to determine which vector can be written as a linear combination of which vectors, and which cannot.

Solution:

Solve the equation

$$x_1 \vec{a}_1 + \dots + x_n \vec{a}_n = \vec{0}$$

then isolate any term \vec{a}_k to one side shows its dependence relationship. If $\vec{x}_k = \vec{0}$, then it shows the vector \vec{a}_k is not a linear combination of other vectors in the set. If $\vec{x}_k = \vec{0}$ for all k , it agrees with the definition of linearly independent set, no vector is a linear combination of other vectors.

2.5 Intro to Linear Transformation

■ Linear Transformation

A transformation (or mapping) T is linear if

- $T[\vec{u} + \vec{v}] = T[\vec{u}] + T[\vec{v}]$ for all \vec{u}, \vec{v} in the domain of T .
- $T[c\vec{u}] = c T[\vec{u}]$ for all \vec{u} and all scalars c .

Theorem: For any linear transformation T , $T[\vec{0}] = \vec{0}$.

(this does not mean that $T[\vec{a}] \neq \vec{0}$ for any $\vec{a} \neq \vec{0}$.)

■ Standard Matrix of Linear Transformation

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T[\vec{x}] = A\vec{x} \text{ for all } \vec{x} \text{ in } \mathbb{R}^n, \text{ and}$$

$$A = \{T[\vec{e}_1], \dots, T[\vec{e}_n]\}$$

where \vec{e}_i is the i th column of the identity matrix I_n .

A is called the standard matrix for the linear transformation T .

(Proof in D.C.Lay 2.6 p.80. Good.)

With this theorem, we could say that every linear transformation from \mathbb{R}^n to \mathbb{R}^m is a matrix transformation, and it's easy to prove that every matrix transformation is linear (see D.C.Lay 2.2 p.52. The one-to-one correspondence of $A\vec{x}$ and $T[x]$ is indicated in the book also. see 2.6, exercise 33, p.85. Easy.)

Note, in this theorem, one may think it will have problems when $n \neq m$. Not so. Because note that $T[\mathbf{e}]$ will return a vector in dimension m .

■ Onto and one-to-one mapping

- A mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be onto \mathbb{R}^m if each \vec{b} in \mathbb{R}^m is the image of at least one \vec{x} in \mathbb{R}^n .
- A mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be one-to-one if each \vec{b} in \mathbb{R}^m is the image of at most one \vec{x} in \mathbb{R}^n .

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T . Then,

- T maps \mathbb{R}^n onto \mathbb{R}^m iff the columns of A span \mathbb{R}^m .
- T is one-to-one iff the columns of A are linearly independent.

3 Matrix Algebra

3.1 Matrix Operations

■ Multiplication of Matrices

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix, then

$$A \otimes B := \{A\vec{b}_1, \dots, A\vec{b}_p\}.$$

Motivaton for the definition:

We want the product $A \otimes B$ to have the property $(AB) \vec{x} = A (B \vec{x})$.

Now,

$$A \cdot (B \cdot \vec{x}) = A \cdot \left(\sum_{i=1}^p x_i \vec{b}_i \right) = \sum_{i=1}^p A \cdot (x_i \vec{b}_i) = \sum_{i=1}^p (x_i (A \cdot \vec{b}_i))$$

Note that $A[x]+A[y]==A[x+y]$.

The **power** of A is defined by

$$A^k := I * \text{Product}[A, \{n\}], \text{ in particular } A^0 = I.$$

By the definition of matrix multiplication, it follows that the
ith row of $(A B) == (\text{ith row of } A) \text{ times (the matrix } B)$

Note that the (ith row of A) is a 1 by n matrix.

■ Transposition of Matrices

The transpose of A is the $n \times m$ matrix, denoted by A^T , whose columns are the corresponding rows of A .

Note: We can also define Transpose as follows: The transpose of a $n \times m$ matrix A is a $m \times n$ matrix A^T such that $\text{Dot}[A \vec{x}, y] == \text{Dot}[A^T y, \vec{x}]$. For example, suppose we have $A := \{\{a,b\},\{c,d\}\}$. Then $\text{Dot}[A \vec{x}, y]$ is then

$$(a x_1 + b x_2) y_1 + (c x_1 + d x_2) y_2$$

To evaluate $\text{Dot}[A^T y, \vec{x}]$, we note that it differs from $\text{Dot}[A \vec{x}, y]$ only by the position of the pairs $\{b,c\}$, $\{x_1,y_1\}$, $\{x_2,y_2\}$ in the final product. So make a switch to get

$$(a y_1 + c y_2) x_1 + (b y_1 + d y_2) x_2$$

Both of them expands to

$$a x_1 y_1 + c x_1 y_2 + b x_2 y_1 + d x_2 y_2$$

This can be expressed and tested in *Mathematica* as

```
Clear[n, m, A, vx, vy];
{n, m} = {3, 5};
A = Table[Random[Integer, {1, 9}], {n}, {m}];
vx = Table[ToExpression["x" <> ToString@i], {i, m}];
vy = Table[ToExpression["y" <> ToString@i], {i, n}];

(A.vx).vy - (Transpose@A.vy).vx // Simplify
```

0

■ Properties of Matrix Multiplication

Let A be $m \times n$ matrix and let B and C have sizes for which the indicated sums and products are defined. Let r be any scalar.

1. $A(BC) = (AB)C$
2. $A(B+C) = AB + AC$
3. $(B+C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
5. $I_m ** A = A = A ** I_n$
6. $(A^T)^T = A$
7. $(A+B)^T = A^T + B^T$
8. $(rA)^T = rA^T$
9. $(AB)^T = B^T A^T$

Proofs in D.C.Lay 3.1. Need to see the proof of 9.

My theorem:

Let A be a square matrix and D be a triangular matrix (a matrix with diagonal values and rest zero.)

$$AD = (DA^T)^T$$

$$DA = (A^T D)^T$$

Note, D is a diagonal matrix. The difference of AD and DA is that DA is like AD except the D comes from above A instead of on the right. Thus it is equal to DA^T , but we need to transpose them back, so the theorem is $AD = (DA^T)^T$.

3.2 The Inverse of a Matrix

■ Inverse of a Matrix

Let A denote a matrix. The inverse of A is a matrix (written as A^{-1}) such that

$$A ** A^{-1} = A^{-1} ** A = I$$

If A^{-1} exists, then we say that A is invertible.

■ inverse of 2x2 matrix

Let $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $d - bc \neq 0$, then A is not invertible, otherwise A is invertible and

$$A^{-1} = \frac{1}{(ad - bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

My Proof:

Suppose the column vectors in A are linearly dependent, so that

$$a = r b$$

$$c = r d$$

for some r . Eliminate r we get $a d - b c = 0$. So if this is true, then column vectors in A are linearly dependent, thus it A has no inverse. If $a d - b c \neq 0$, then the column vectors are not linearly dependent, so an inverse exists. In particular, it means the following two systems have solution:

$$a x_1 + b x_2 = 1$$

$$c x_1 + d x_2 = 0$$

and

$$a y_1 + b y_2 = 0$$

$$c y_1 + d y_2 = 1$$

We solve them by algebra to complete the proof.

Question: Is there a square matrix M or N such that $M A = A^T$ or $A N = A^T$ for any square matrix A ?

Answer: Not always, for example the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. If M or N exists, then

$$M = A^T A^{-1}, N = A^{-1} A^T.$$

End of example.

Question: We may think of transposition as a transformation of a one-to-one function of matrix, and the fixed point of this transformation is all the identity matrices. What is the geometric significance of transposition?

To Do:

- 1) Show that identity matrix is unique. i.e. if $E A = A$ or $A E = A$, then $E = I$.
- 2) Show, or find an example such that $M B = N B$, but $M \neq N$; $B M = B N$ but $M \neq N$.
- 3) Show that for any matrix A , C such that $A C = I$, then $C A = I$.

■ Theorem of Inverses

- If A is an invertible $n \times n$ matrix, then for each b in \mathbb{R}^n , the equation $A \vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1} b$.
- If A is an invertible matrix, then
 1. $(A^{-1})^{-1} = A$
 2. $(A^T)^{-1} = (A^{-1})^T$
- If A and B are $n \times n$ invertible matrices, then $(A B)$ is also invertible, and $(A B)^{-1} = B^{-1} A^{-1}$
- An $n \times n$ matrix A is invertible iff A is row equivalent to I_n , and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n to A^{-1} .

To Do: Read the proof of this.

My proof of one of the above (the rest is in the book):

$$(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$$

$$(\mathbf{A}^T) (\mathbf{A}^T)^{-1} = (\mathbf{A}^T) (\mathbf{A}^{-1})^T$$

$$\mathbf{I} = (\mathbf{A}^T) (\mathbf{A}^{-1})^T$$

$$\mathbf{I}^T = ((\mathbf{A}^T) (\mathbf{A}^{-1})^T)^T$$

$$\mathbf{I} = ((\mathbf{A}^{-1})^T)^T (\mathbf{A}^T)^T$$

$$\mathbf{I} = \mathbf{I}$$

(this proof may be wrong, because it assumed $(\mathbf{A}^T)^{-1}$ exist outright.)

3.3 Characterizations of Invertible Matrices

■ The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. The following statements are equivalent.

- A is an invertible matrix.
- A is row equivalent to the $n \times n$ identity matrix.
- A is a product of elementary matrices.
- The equation $A \vec{x} = \vec{0}$ has only the Zero-solution.
- The equation $A \vec{x} = \vec{b}$ has at least one solution for any b in \mathbb{R}^n . The equation $A \vec{x} = \vec{b}$ has a unique solution for any b in \mathbb{R}^n .
- The columns of A form a linearly independent set.
- The columns of A span \mathbb{R}^n .
- The linear transformation $T[\vec{x}] := A \vec{x}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is one-to-one and onto.
- There is an $n \times n$ matrix C such that $AC = I$.
- There is an $n \times n$ matrix D such that $DA = I$.
- A^T is an invertible matrix.
- The columns of A form a basis of \mathbb{R}^n .
- $\text{Col}[A] = \mathbb{R}^n$.
- $\text{Dim}[\text{Col}[A]] = n$.
- $\text{Rank}[A] = n$
- $\text{Nul}[A] = \{\vec{0}\}$.
- $\text{Dim}[\text{Nul}[A]] = 0$.
- The number 0 is not an eigenvalue of A .

5 Vector spaces

5.1 Vector Spaces and Subspaces

■ Vector Space definition

A Vector Space is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real or complex), subject to the ten axioms listed below. The axioms must hold for all vectors \vec{u} , \vec{v} , and \vec{w} in V and for all scalars c and d .

1. The sum of \vec{u} and \vec{v} , denoted by $\vec{u} + \vec{v}$, is in V .
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.
4. There is a Zero vector $\vec{0}$ in V such that $\vec{u} + \vec{0} = \vec{u}$.
5. For each \vec{u} in V there is a vector $-\vec{u}$ in V such that $\vec{u} + (-\vec{u}) = \vec{0}$.
6. The scalar multiple of \vec{u} by c , denoted by $c\vec{u}$, is in V .
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$.
8. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$.
9. $c(d\vec{u}) = (cd)\vec{u}$.
10. $1\vec{u} = \vec{u}$.

■ Alternative definition of a vector space

Let K be a division ring. A vector space over K is an ordered triple $\{E, +, \cdot\}$ such that $\{E, +\}$ is an abelian group and \cdot is a function from $K \times E$ into E satisfying

- $c \cdot (\vec{u} + \vec{v}) = c \cdot \vec{u} + c \cdot \vec{v}$
- $(c + d) \cdot \vec{u} = c \cdot \vec{u} + d \cdot \vec{u}$
- $(c \cdot d) \cdot \vec{u} = c \cdot (d \cdot \vec{u})$
- $1 \cdot \vec{u} = \vec{u}$

for all $\vec{u}, \vec{v} \in E$ and all $c, d \in K$. Elements of E are called vectors, elements of K are called scalars, and \cdot is called scalar multiplication.

Similarly, $\{\mathbb{R}^n, +, \cdot\}$ is a vector space over \mathbb{R} for each $n \in \mathbb{N}$.

■ Corollaries

The zero vector $\vec{0}$ in any vector space is unique, and the vector $-\vec{u}$ is unique for each \vec{u} in V , and

1. $0 \cdot \vec{u} = \vec{0}$
2. $c \cdot \vec{0} = \vec{0}$
3. $-\vec{u} = (-1) \cdot \vec{u}$

My Proof of $-u = (-1) \cdot u$.

$\vec{0} = 0 \cdot u$ (corollary 1)

$0 \cdot u = (1-1) \cdot u$ (property of arithmetic)

$(1-1) \cdot u = 1 \cdot u + (-1) \cdot u$ by axiom 8.

$1 \cdot u + (-1) \cdot u = u + (-1) \cdot u$ by axiom 10. Connection previous equation to get

$u + (-1) \cdot u = \vec{0}$

$(-u) + u + (-1) \cdot u = \vec{0} + (-u)$ Add $(-u)$ to both sides.

$\vec{0} + (-1) \cdot u = -u$ by axiom 4 and 5.

$(-1) \cdot u = -u$

■ Subspace of a Vector Space

A subspace of a vector space V is a subset H of V such that H is itself a vector space under the same operations of addition and scalar multiplication that are already defined on V .

■ Subspace Test

A subset H of vector space V is a subspace of V iff the following conditions are all satisfied:

1. The Zero vector of V is in H .
2. If \vec{u} and \vec{v} are in H , then $\vec{u} + \vec{v}$ is in H .
3. If \vec{u} is in H and c is any scalar, then $c \vec{u}$ is in H .

■ Spanning Set and Subspace

Given $\vec{a}_1, \dots, \vec{a}_n$ in a vector space V , the set $\text{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$ is a subspace of V .

We call $\text{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$ the subspace generated by $\{\vec{a}_1, \dots, \vec{a}_n\}$. Given any subspace H of V , a spanning set for H is a set $\{\vec{a}_1, \dots, \vec{a}_n\}$ in H such that $H = \text{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$.

5.2 Null Spaces, Column Spaces, and Linear Transformations

■ Null Space

Given an $m \times n$ matrix A , the null space of A (denoted $\text{Nul}[A]$) is the set of all solutions to the homogeneous equation $A \cdot \vec{x} = \vec{0}$. In set notation,

$$\text{Nul}[A] := \{\vec{x}: \vec{x} \text{ is in } \mathbb{R}^n \text{ and } A \cdot \vec{x} = \vec{0}\}$$

$\text{Nul}[A]$ is the set of all \vec{x} in \mathbb{R}^n that are mapped into the zero vector of \mathbb{R}^m by the linear transformation $A \cdot \vec{x}$.

■ Column Space

Given an $m \times n$ matrix A , the column space of A (denoted $\text{Col}[A]$) is the set of all linear combinations of the columns of A . In set notation,

$$\text{Col}[A] := \{\vec{b}: \vec{b} \text{ is in } \mathbb{R}^m \text{ and } \vec{b} = A \cdot \vec{x} \text{ for some } \vec{x} \text{ in } \mathbb{R}^n\}$$

Column space is just the range of a linear transformation.

The null space of an $m \times n$ matrix A is a subspace of \mathbb{R}^n .

The column space of an $m \times n$ matrix A is a subspace of \mathbb{R}^m .

When talking about linear transformation, null space and column space are often called kernel and range respectively. Kernel is the set of \vec{x} such that $T[\vec{x}] = \vec{0}$, and range has the usual meaning for a function -- all possible outputs.

5.3 Linearly Independent Sets; Bases

■ definition: Linearly Independence

A Set $\{\vec{v}_1, \dots, \vec{v}_n\}$ of two or more vectors, with $\vec{v}_1 \neq \vec{0}$, is linearly dependent iff some \vec{v}_j (with $j > 1$) is a linear combination of the preceding vectors.

We require $\vec{v}_1 \neq \vec{0}$ because, suppose we have an ordered set $S := \{\vec{0}, \vec{u}\}$. This is a linearly dependent set by definition, but there are no \vec{v}_j (with $j > 1$) that is a linear combination of the preceding vectors, $\vec{v}_1, \dots, \vec{v}_{j-1}$; in other words, $\vec{u} \neq c \vec{0}$. On the other hand, if $S := \{\vec{u}, \vec{0}\}$, we have $\vec{0} = 0 \vec{u}$, thus showing S is linearly dependent as the theorem dictates.

Question: why can't the definition be that "there are no vector that is a linear combination of two or more vectors in the set?"

■ definition: Basis of a Vector Space

Let H be a subspace of a vector space V . Let B be a set of vectors in V . B is a basis for H if

1. B is a linearly independent set.
2. the subspace spanned by B is equal to H .

Note: the set $\{\vec{0}\}$ do not have a basis because $\{\vec{0}\}$ is dependent by definition.

• Let $\vec{e}_1, \dots, \vec{e}_n$ be the columns of the $n \times n$ identity matrix I_n . The set $\{\vec{e}_1, \dots, \vec{e}_n\}$ is called the standard basis for \mathbb{R}^n .

■ The Spanning Set Theorem

Let $S := \{\vec{v}_1, \dots, \vec{v}_n\}$ be a set in vector space.

- Suppose \vec{v}_k in S is a linear combination of the remaining vectors, and S_2 is a set formed by removing the element \vec{v}_k from S , then $\text{Span}[S] = \text{Span}[S_2]$.
- If $\text{Span}[S] \neq \{\vec{0}\}$, some subset of S is a basis for $\text{Span}[S]$.

Note: S do not have a basis if $S := \{\vec{0}\}$. That's why we require $\text{Span}[S] \neq \{\vec{0}\}$ in the theorem.

■ Basis of Col[A]

The pivot columns of a matrix A forms a basis for $\text{Col}[A]$.

The following theorem is "my own". It's incomplete. I need to complete the theorem statement and proof.

Let V and W be vector spaces, let $T: V \rightarrow W$ be a linear transformation, and let $S := \{\vec{v}_1, \dots, \vec{v}_k\}$ be a subset of V , and $S_2 := \{T[\vec{v}_1], \dots, T[\vec{v}_k]\}$.

- If S is linearly dependent in V , then S_2 is linearly dependent in W .
- If S_2 is linearly independent in W , then S is linearly independent in V . ???

If T is one-to-one transformation, then

- S is linearly dependent iff S_2 is linearly dependent.
- S is linearly independent iff S_2 is linearly independent.

My Proof:

Suppose S is linearly dependent. Whether S_2 is linearly dependent depends on whether $c_1 = c_2 = \dots = c_k = 0$ is the only solution to the equation

$$c_1 T[v_1] + c_2 T[v_2] + \dots + c_k T[v_k] = \vec{0} \text{ iff}$$

$$T[c_1 v_1] + T[c_2 v_2] + \dots + T[c_k v_k] = \vec{0} \text{ iff}$$

$$T[c_1 v_1 + c_2 v_2 + \dots + c_k v_k] = \vec{0} \text{ iff}$$

$$(c_1 v_1 + c_2 v_2 + \dots + c_k v_k = \vec{0} \text{ or } T[b] = \vec{0} \text{ for some } b \neq \vec{0})$$

The theorems above easily follows from here.

A vector equation of the form $c_1 v_1 + c_2 v_2 + \dots + c_k v_k = b$ has either one solution, infinitely many solution, or no solution. Need to see a proof. To Do.

5.4 Coordinate Systems

■ theorem: The Unique representation

Let $B := \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ be a basis for a vector space V . Then for each \vec{x} in V , there exists a unique set of scalars c_1, \dots, c_n , such that $\vec{x} = c_1 \vec{b}_1 + \dots + c_n \vec{b}_n$.

Proof:

Suppose there are two representations $x = \sum [c_i * b_i, \{i, n\}]$ and $x = \sum [d_i * b_i, \{i, n\}]$.

Subtract the equations we have

$$\text{zeroVector} = \sum [(c_i - d_i) * b_i, \{i, n\}]$$

Since all b_i are linearly independent, thus all $(c_i - d_i)$ must be 0, which means $c_i = d_i$ for all i .

■ Definition: Coordinates

Suppose the set $B := \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ is a basis for V and \vec{x} is in V . The Coordinates of \vec{x} relative to the basis B are the weights c_1, \dots, c_n such that $\vec{x} = c_1 \vec{b}_1 + \dots + c_n \vec{b}_n$.

■ $\vec{x} \mapsto \{\vec{x}\}_B$ is one-to-one and onto

Let $B := \{\vec{b}_1, \dots, \vec{b}_n\}$ be an ordered basis for a vector space V . Then the coordinate mapping $\vec{x} \mapsto \{\vec{x}\}_B$ is a one-to-one linear transformation from V onto \mathbb{R}^n .

5.5 The Dimension of a Vector Space

■ Number of Vectors in Basis

- If a vector space V has a basis $\mathcal{B} := \{\vec{b}_1, \dots, \vec{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.
- If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

■ Dimension of V

If V is spanned by a finite set, then V is said to be finite-dimensional, and the dimension of V , written as $\dim[V]$, is the number of vectors in a basis for V . The dimension of zero vector space $\{\vec{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be infinite-dimensional.

■ Linear Dependence, Span, Dimension and Basis

- Let H be a subspace of a finite-dimensional vector space V . Any linearly independent set in H can be expanded, if necessary, to a basis for H . Also, H is finite-dimensional and $\dim[H] \leq \dim[V]$.
- Let V be an n -dimensional vector space, $n \geq 1$, and let S be a subset of V that contains exactly n elements. Then:
 - * If S is linearly independent, then S is a basis for V .
 - * If S spans V , then S is a basis for V .

The dimension of $\text{Nul}[A]$ is the number of free variables in the equation $A \cdot \vec{x} = \vec{0}$, and the dimension of $\text{Col}[A]$ is the number of pivot columns in A .

5.6 Rank

■ Basis for Row Spaces

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as B .

■ Rank

The rank of A is the dimension of the column space of A .

The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A , also equals the number of pivot positions in A and satisfies the equation

$$\text{Rank}[A] + \text{Dim}[\text{Nul}[A]] = n$$

6 Eigenvectors and Eigenvalues

6.1 Eigenvectors and Eigenvalues

■ Eigenvector and Eigenvalue

An eigenvector of an $n \times n$ matrix A is a nonzero vector \vec{x} such that $A \cdot \vec{x} = \lambda * \vec{x}$ for some scalar λ . A scalar λ is called an eigenvalue of A if there is a vector $\vec{x} \neq \vec{0}$ such that $A \cdot \vec{x} = \lambda * \vec{x}$; such an \vec{x} is called an eigenvector corresponding to λ .

Eigenspace:

Suppose $A \cdot \vec{x} = \lambda \vec{x}$, then

$$A \cdot \vec{x} - \lambda \vec{x} = \vec{0}$$

$$(A - \lambda I) \cdot \vec{x} = \vec{0}$$

The null space of $(A - \lambda I)$ is called the eigenspace of A corresponding to λ .

■ triangular matrix and eigenvalue

Let A be a triangular matrix. Then the eigenvalues of A are the entries on its main diagonal.

Proof is easy. Look at the matrix $(A - \lambda I)$. The column vectors in this matrix must be linearly independent, and the only way is for λ to equal to some of the diagonal entries.

■ Linear independence of engine vectors

If $\vec{v}_1, \dots, \vec{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A , then the set $\{\vec{v}_1, \dots, \vec{v}_r\}$ is linearly independent.

Proof:

If $\{\vec{v}_1, \dots, \vec{v}_r\}$ is linearly dependent, then there is a least index p such that \vec{v}_{p+1} is a linear combination preceding (linearly independent) vectors, and there exist scalars c_1, \dots, c_n such that

$$\sum_{i=1}^p c_i \vec{v}_i = \vec{v}_{p+1} \quad (*5*)$$

Multiplying both sides by A and using the fact that $A \cdot \vec{v}_k = \lambda_k \vec{v}_k$ for each k , we obtain

$$\sum_{i=1}^p c_i A \cdot \vec{v}_i = A \cdot \vec{v}_{p+1}$$

$$\sum_{i=1}^p c_i \lambda_i \vec{v}_i = \lambda_{p+1} \vec{v}_{p+1} \quad (*6*)$$

Multiplying both sides of (*5*) by λ_{p+1} and subtracting the result from (*6*), we have

$$\sum_{i=1}^p c_i (\lambda_i - \lambda_{p+1}) \vec{v}_i = \vec{0} \quad (*7*)$$

Since $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly independent, the weights in (*7*) are all zero. But none of the factors $(\lambda_i - \lambda_{p+1})$ are zero, because the eigenvalues are distinct. Hence $c_i = 0$ for $i=1, \dots, p$. But then (*5*) says that $\vec{v}_{p+1} = \vec{0}$, which is impossible. Hence $\{\vec{v}_1, \dots, \vec{v}_r\}$ cannot be linearly dependent and therefore must be linearly independent.

6.2 The Characteristic Equation

■ Determinant

One way to define determinants is as follows: For an $n \times n$ matrix. We row reduce it to echelon form using the following methods:

- * Replace a row by the linear combination of other rows.
- * Interchange a row with another row.

(in particular, scaling of a row by itself is not allowed.)

The determinant of A is the product of the diagonal entries times $(-1)^r$, where r is the number of times we interchanged rows.

If the matrix is not invertible, it will contain a 0 in its diagonal, and thus the determinant will be 0.

If A is a 3×3 matrix, then $\text{Abs}[\text{Det}[A]]$ is the volume of the parallelepiped defined by the column vectors of A .

■ Properties of Determinants

Let A and B be an $n \times n$ matrices.

- A is invertible iff $\text{Det}[A] \neq 0$
- $\text{Det}[A \cdot B] = \text{Det}[A] \cdot \text{Det}[B]$
- $\text{Det}[A^T] = \text{Det}[A]$
- If A is triangular, then $\text{Det}[A]$ is the product of the entries on the main diagonal of A .
- A row replacement operation on A does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

Proofs in D.C.Lay, chapter 4.

■ characteristic polynomial

The scalar equation $\text{Det}[A - \lambda I] = 0$ is called the characteristic equation of A .

A scalar λ is an eigenvalue of an $n \times n$ matrix A iff λ satisfies the characteristic equation $\text{Det}[A - \lambda I] = 0$.

This is so because:

$$A \cdot x = \lambda \cdot x$$

$$A \cdot x - \lambda \cdot x = 0$$

$$A \cdot x - \lambda \cdot I \cdot x = 0$$

$$(A - \lambda \cdot I) \cdot x = 0.$$

■ similar matrixes

If A and B are $n \times n$ matrices, then A is similar to B if there is an invertible matrix P such that $P^{-1} \cdot A \cdot P = B$. (which implies $A = P \cdot B \cdot P^{-1}$)

Xah's note: I think: in group theory, this idea is called conjugate class. i.e. two elements a and b are in the same conjugate class if there exist an element p , such that $p^{-1} \cdot a \cdot p = b$.

theorem: similar matrixes has the same eigensystem

If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues.

Proof: If $B = P^{-1} \cdot A \cdot P$, then

$$B - \lambda \cdot I = P^{-1} \cdot A \cdot P - \lambda \cdot P^{-1} \cdot P = P^{-1} \cdot (A - \lambda \cdot I) \cdot P$$

using the multiplicative property of determinants, we have

$$\begin{aligned} \text{Det}[B - \lambda \cdot I] &= \text{Det}[P^{-1} \cdot (A - \lambda \cdot I) \cdot P] = \text{Det}[P^{-1}] \cdot \text{Det}[(A - \lambda \cdot I)] \cdot \text{Det}[P] \\ &= \text{Det}[P^{-1}] \cdot \text{Det}[P] \cdot \text{Det}[(A - \lambda \cdot I)] = \text{Det}[(A - \lambda \cdot I)] \end{aligned}$$

Some common algorithms for estimating the eigenvalues of a matrix is based on the above theorem. They include QRDecomposition and Jacobi's method. See p.287 of David C.Lay.

■ Difference equations and dynamic systems

Eigensystem is the key to difference equations and dynamic systems. For example, we are given the vector sequence

$$\begin{aligned} f[0] &:= \vec{a} \\ f[n] &:= A \cdot f[n-1] \end{aligned}$$

This sequence can be written as

$$f[n] := A^n \cdot \vec{a}$$

One can avoid the multiplication of matrices by using eigensystems. Suppose A is 2×2 and we have the eigensystems $\{\{\lambda_1, \lambda_2\}, \{\vec{u}_1, \vec{u}_2\}\}$.

Remember that $A^n \cdot x = \lambda^n \cdot x$ for eigensystem λ and x .

Write \vec{a} in terms of eigenvectors (this can be done since the eigenvectors forms a basis). Suppose we have $\vec{a} = c_1 \vec{u}_1 + c_2 \vec{u}_2$

$$\text{Now, } f[n] = A^n \cdot \vec{a} = A^n \cdot (c_1 \vec{u}_1 + c_2 \vec{u}_2) = c_1 A^n \cdot \vec{u}_1 + c_2 A^n \cdot \vec{u}_2 = c_1 \lambda_1^n \vec{u}_1 + c_2 \lambda_2^n \vec{u}_2$$

thus

$$f[n] = c_1 \lambda_1^n \vec{u}_1 + c_2 \lambda_2^n \vec{u}_2$$

This is a closed form of $f[n]$. The closed form facilitate computation and analysis of the behavior of $f[n]$. We may use it to find $\text{Limit}[f[n], n \rightarrow \text{Infinity}]$.

6.3 Diagonalization

■ definition: Diagonalization

A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix, that is, if $A = P \cdot D \cdot P^{-1}$ for some invertible matrix P and some diagonal matrix D .

■ theorem: Diagonalization

An $n \times n$ matrix A is diagonalizable iff A has n linearly independent eigenvectors.

If $A = P \cdot D \cdot P^{-1}$ where D is diagonal, then the diagonal entries of D are eigenvalues of A and the columns of P are the corresponding eigenvectors.

Proof:

Let u_1, u_2, \dots be independent eigenvectors of A , and $\lambda_1, \lambda_2, \dots$ be the corresponding eigen values.

We have

$$[A \cdot u_1 \ A \cdot u_2 \ \dots] = [\lambda_1 \cdot u_1 \ \lambda_2 \cdot u_2 \ \dots] \quad (*1*)$$

which can be rewritten as

$$A \cdot P = P \cdot D$$

by the given definition of matrix P and D . Since u_i are independent, P^{-1} exist. Right multiply both sides by P^{-1} we have

$$A = P \cdot D \cdot P^{-1}$$

This proves the half of first part of the theorem. Now, if given $A = P \cdot D \cdot P^{-1}$ for some P, D , we can use $(*1*)$ to show that $A \cdot u_i = \lambda_i \cdot u_i$ by equating columns. Now, since u_i for all i are independent and none are zero vectors, it shows that λ_i are eigenvalues corresponding to the eigenvectors u_i .

The second part of the theorem is also proven in the process.

End of proof.

■ theorem

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \dots, \lambda_p$. For $k=1, \dots, p$, let B_k be a basis for the eigenspace corresponding to λ_k . Let B be the union of B_1, \dots, B_p . Then B is linearly independent.

Notes: recall that eigenspace of an eigenvalue λ is defined to be all eigenvectors that has λ as their eigenvalue. That is to say, an eigenvalue may have more than one linearly independent eigenvectors. The theorem does not say an $n \times n$ matrix will always have n eigenvectors. It only says that if A has n linearly independent eigenvectors, then A is diagonalizable.

Proof: (need editing)

Suppose $\lambda_1, \dots, \lambda_n$ are the distinct eigenvalues of matrix A . Suppose B_i are eigenvector bases corresponding to λ_i and each B_i has dimensions $\text{Length}[B_i]$.

Let s_i be $:= \text{Sum}[c_{i_j} \cdot B_i[j], \{j, \text{Length}[B_i]\}]$.

surfeit to say that if $\text{Sum}[s_i, \{i, n\}] = \text{zeroVector}$, then $s_i = 0$ for all i . This is so because eigenspace of different eigenvalues has no intersection. Thus, the above implies all $c_{i_j} = 0$ in

$$\text{Sum}[c_{i_j} \cdot B_i[j], \{j, \text{Length}[B_i]\}] = \text{zeroVector}$$

Which implies all $c_{i_j} = 0$ in

$$\text{Sum}[(c_{i_j} \cdot B_i[j]), \{i, n\}, \{j, \text{Length}[B_i]\}] = \text{zeroVector}$$

Thus B is linearly independent.

are all zero.

We know that $c_j=0$ in $\text{Sum}[(c_j), B_i[[j]], \{j, n\}] == \text{zeroVector}$.

$\text{Sum}[\text{Sum}[(c_{i_j} * B_i[[j]]), \{j, \text{Length}[B_i]\}], \{i, n\}]$

Suppose b_{i_m} is an eigenvector in B_i .

$\text{Sum}[\text{Plus}@@(c_{i_j} * B_i), \{i, n\}] == \text{zeroVector}$

and all c_{i_j} must be zero.

This means each of $\text{Plus}@@(c_{i_j} * B_i) == \text{zeroVector}$.

6.4 Eigenvectors and Linear Transformation

■ theorem: Diagonal Matrix Representation

Suppose $A := P.D.P^{-1}$, where D is a diagonal $x \times n$ matrix. If \mathcal{B} is the basis for \mathbb{R}^n formed from the columns of P , then D is the \mathcal{B} -matrix of the transformation $\vec{x} \mapsto A.\vec{x}$.

Proof: (verbatim from the book. Needs editing.)

Denote the columns of P by b_1, \dots, b_n , so that $B = \{b_1, \dots, b_n\}$, and $P = [b_1 \dots b_n]$. In this case P is the change-of-coordinates matrix $P_{\mathcal{B}}$ discussed in Section 5.4, where

$P[X]_{\mathcal{B}} = x$ and $[x]_{\mathcal{B}} = (P^{-1}).x$

If $T[x] = A.x$ for x in \mathbb{R}^n , then

$[T]_{\mathcal{B}} = [[T[b_1]]_{\mathcal{B}} \dots [T[b_n]]_{\mathcal{B}}]$

$== [[A.b_1]_{\mathcal{B}} \dots [A.b_n]_{\mathcal{B}}]$

$== [(P^{-1}).A.b_1 \dots (P^{-1}).A.b_n]$

$== (P^{-1}).A.[b_1 \dots b_n]$

$== (P^{-1}).A.P$

Since $A = P.D.(P^{-1})$, we have $[T]_{\mathcal{B}} = (P^{-1}).A.P = D$.

6.5 Complex Eigenvalues

linear map of an abstract vector space (xah's note)

To define a linear map T , we don't have to know the image of every element in V . We only need to know the image of a set of vectors that is a basis for V . This is so because the property of linear map that $L[a+b] = L[a] + L[b]$ and $L[\alpha*a] = \alpha*L[a]$. In particular, every vector can be written as linear combinations of the basis vectors, thus we can find their image. (Greek letter denote scalars.)

It is extremely convenient to find the isomorphism between an abstract linear space and the linear space in \mathbb{R}^n , because the latter can be computed more easily, and linear map can be written as matrixes.

The following describes the method of finding an isomorphic vector space in \mathbb{R}^n from a given abstract vector space.

Given a n dimensional linear space $\{V, +, *\}$ over \mathbb{R}^1 . Suppose B is a basis. A linear mapping T is defined by specifying the images of basis vectors. i.e., we know the value of $T[b[k]]$ for $1 \leq k \leq n$. Let us use B as the coordinate basis. We want to find the matrix A corresponding to T . Let $\{b[k]\}$ denote the coordinates of $b[k]$ with respect to basis B . Let $\{e[k]\}$ denote the standard basis, that is: $\{e[1]\} = \{1, 0, 0, \dots\}$, $\{e[2]\} = \{0, 1, 0, 0, \dots\}$, etc. The coordinates of $b[k]$ with respect to B is $\{e[k]\}$, i.e. $\{b[k]\} = \{e[k]\}$.

Now, the coordinates of $T[\{b[k]\}]$ can be denoted as $\{T[\{b[k]\}]\}$.

Let x be a vector in \mathbb{R}^n . $x[i]$ be its i th coordinates. We can write x as $\text{Sum}[x[i] * \{e[i]\}, \{i, 1, n\}]$. Now,

$$T[x] = T[\text{Sum}[x[i] * \{e[i]\}, \{i, 1, n\}]] = \text{Sum}[x[i] * T[\{e[i]\}], \{i, 1, n\}]$$

Thus we see that k th column of matrix A is $\{T[\{b[k]\}]\}$.

7 Orthogonality and Least-squares

7.1 Inner Product, Length, And Orthogonality

■ Inner product definition

The inner product of vectors u and v in \mathbb{R}^n is defined to be $\text{Sum}[u[i] * v[i], \{i, 1, n\}]$, where $u[i]$ and $v[i]$ are the elements in u and v .

■ Properties of inner product

Let u , v , and w be vectors in \mathbb{R}^n , and let c be a scalar.

$$u \cdot v = v \cdot u$$

$$(u + v) \cdot w = u \cdot w + v \cdot w$$

$$(c * u) \cdot v = c(u \cdot v) = u \cdot (c \cdot v)$$

$$u \cdot u \geq 0, \text{ and } u \cdot u = 0 \text{ iff } u = \vec{0}.$$

■ Length of a vector

The length (or norm) of \vec{v} is the nonnegative scalar $\|\vec{v}\|$ defined by $\|\vec{v}\| := \sqrt{\vec{v} \cdot \vec{v}}$.

■ Distance

For \vec{u} and \vec{v} in \mathbb{R}^n , the distance between them is defined to be $\|\vec{u} - \vec{v}\|$.

■ Orthogonality

Two vectors u and v in \mathbb{R}^n are orthogonal (to each other) if $u \cdot v = 0$.

The motivation for this definition:

We want to look at vectors as lines in Euclidean space, and define orthogonality as being perpendicularity. Suppose u and v are vectors. They are orthogonal if the distance from u to v and u to $-v$ are equal. Using Pythagorean theorem, the distance from u to v is

$$\|u-v\|^2 = \|u\|^2 + \|v\|^2 - 2u \cdot v$$

The distance from u to $-v$ is

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2u \cdot v$$

we see that they are equal iff $u \cdot v = 0$.

■ Pythagorean Theorem

Two vectors u and v are orthogonal iff $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

(this follows directly from definitions)

■ Orthogonal Complements

Let W be a subspace of \mathbb{R}^n , let z be a vector. If z is orthogonal to every vector in W , then z is said to be orthogonal to W . The set of vectors that are orthogonal to W is called the orthogonal complement of W , written as W^\perp or $\perp[W]$.

■ theorem:

A vector v is in $\perp[W]$ iff v is orthogonal to all vectors in a set that spans W .

Proof:

Let $\{a_1, \dots, a_n\}$ be a set of basis vectors for a subspace W of \mathbb{R}^n .

Given: v is orthogonal to a_i for all i . This means $a_i \cdot v = 0$ for all i .

We want to prove that v is orthogonal to any linear combinations of a_i . This means

$$\sum_{i=1}^n c_i a_i \cdot v = 0$$

where c_i are scalars. By properties of inner product, the equation is equivalent to

$$\sum_{i=1}^n (c_i a_i) \cdot v = 0$$

$$\sum_{i=1}^n c_i (a_i \cdot v) = 0$$

$$\sum_{i=1}^n c_i \cdot 0 = 0$$

This proves half of the first statement.

If $b \cdot v = 0$ for all b in W , then it means

$$\sum_{i=1}^n (c_i a_i) \cdot v = 0$$

by inner product property,

$$c_i a_i \cdot v = 0 \text{ for all } i.$$

$$a_i \cdot v = 0$$

End of proof.

■ dimensions of $\perp[W]$

Let W be any vector space. The dimension of $\perp[W]$ is 1.

Proof:

Let x be a non-zero vector in W . Suppose b_1, b_2 are basis vectors in $\perp[W]$, thus $\perp[W]$ has dimension 2.

It follows that $b_1 \cdot x = 0$, $b_2 \cdot x = 0$, and $(b_1 + b_2) \cdot x = 0$. Combine the equations we have

$$(b_1 + b_2) \cdot x = b_1 \cdot x$$

For vectors u_1, u_2, v , $u_1 \cdot v = u_2 \cdot v$ iff $u_1 = u_2$, thus

$$b_1 + b_2 = b_1$$

$$b_2 = \text{zeroVector}$$

but this cannot be because basis vectors cannot be zero vectors. Thus, b_1 and b_2 cannot be a basis for $\perp[W]$, and $\perp[W]$ cannot have dimension 2. Similarly, if b_1, b_2, b_3 are basis for $\perp[W]$, we can show that $(b_1 + b_2 + b_3) = b_1$, $b_2 + b_3 = \text{zeroVector}$, and this cannot be because b_2 and b_3 are independent. Similarly, $\perp[W]$ cannot have dimensions greater than 1.

■ Theorem: Orthogonality, row space, null space, and column space

Let A be an $m \times n$ matrix. Then the orthogonal complement of the row space of A is the nullspace of A , and the orthogonal complement of the column space of A is the nullspace of $\mathcal{T}[A]$. In symbols,

$$\perp @ \text{Row}[A] = \text{Nul}[A], \quad \perp @ \text{Col}[A] = \mathcal{T} @ \text{Nul}[A]$$

■ Proof

The row-column rule for computing $A \cdot x$ shows that if x is in $\text{Nul } A$, then x is orthogonal to each row of A (with the rows treated as vectors in \mathbb{R}^n). Since the rows of A span the row space, x is orthogonal to $\text{Row } A$. Conversely, if x is orthogonal to $\text{Row } A$, then x is certainly orthogonal to the rows of A , and hence $A \cdot x = 0$. This proves the first statement. The second statement follows from the first by replacing A with $\mathcal{T}[A]$ and using the fact that $\text{Col}[A] = \mathcal{T} @ \text{Row}[A]$.

7.2 Orthogonal Sets

■ orthogonal set and linear independence

A set of vectors $\{u_1, \dots, u_p\}$ in \mathbb{R}^n is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is, if $u_i \cdot u_j = 0$ whenever $i \neq j$.

If $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ is an orthogonal set of nonzero vectors in \mathbb{R}^n , then S is linearly independent.

Proof:

We want to show that the only solution to the equation $\vec{0} = \sum_{i=1}^p c_i \vec{v}_i$ is for $c_i = 0$ for all i . Multiply both sides by \vec{v}_1 to obtain

$$0 = \vec{0} \cdot \vec{v}_1 = \left(\sum_{i=1}^p c_i * \vec{v}_i \right) \cdot \vec{v}_1$$

by distribution property, we have

$$0 = \left(\sum_{i=1}^p c_i * (\vec{v}_i \cdot \vec{v}_1) \right)$$

Since elements of S are pairwise orthogonal, their dot product is 0, thus we have

$$0 = c_1 * (\vec{v}_1 \cdot \vec{v}_1)$$

which shows that c_1 is 0. Similarly, all other scalars must be 0. End of Proof.

■ computing coordinates wrt a given orthogonal basis

Let $\{\vec{v}_1, \dots, \vec{v}_p\}$ be an orthogonal basis for a subspace W of \mathbb{R}^n . Let \vec{y} be a vector in W . The coordinates of \vec{y} with respect to the basis is $\text{Table} \left[\left(\frac{\vec{y} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i} \right), \{i, 1, p\} \right]$.

Proof:

we want to show that $c_i = (\vec{y} \cdot \vec{v}_i) / (\vec{v}_i \cdot \vec{v}_i)$ is a solution to the equation $\vec{y} = \sum_{i=1}^p c_i * \vec{v}_i$.

To solve the equation, dot both sides by \vec{v}_1 , we get

$$\vec{y} \cdot \vec{v}_1 = (c_1 * \vec{v}_1) \cdot \vec{v}_1$$

(most vector terms are eliminated because their dot product is zero)

Isolate c_1 to solve one coordinate. Other coordinates can be solved similarly.

■ orthonormal set

A set $\{u_1, \dots, u_p\}$ is an orthonormal set if it is an orthogonal set of unit vectors.

An $m \times n$ matrix U has orthonormal columns iff $(\mathcal{T} @ U).U = I$.

Proof: (See D.C.Lay)

The proof for the general matrix is tedious, though not much insight. Essentially, the steps are to carry out the product showing each matrix entry in detail, and show that in order for the columns in U to be orthonormal, certain product of elements must be 0 or 1, and when this occurs, the whole thing equals to the identity matrix.

■ length and orthogonality invariance

Let U be an $m \times n$ matrix with orthonormal columns, and let \vec{x} and \vec{y} be in \mathbb{R}^n . Then

(a) $\|U \cdot \vec{x}\| = \|\vec{x}\|$

(b) $(U \cdot \vec{x}) \cdot (U \cdot \vec{y}) = \vec{x} \cdot \vec{y}$

(c) $(U \cdot \vec{x}) \cdot (U \cdot \vec{y}) = 0$ iff $\vec{x} \cdot \vec{y} = 0$.

Property (a) and (c) say that the linear mapping $\vec{x} \mapsto U \cdot \vec{x}$ preserves lengths and orthogonality. Proof should be easy.

7.3 Orthogonal projection

■ The orthogonal Decomposition theorem

Let W be a subspace of \mathbb{R}^n that has an orthogonal basis. Then each \vec{y} in \mathbb{R}^n can be written uniquely in the form

$$\vec{y} = \vec{a} + \vec{z} \quad \langle\langle 1 \rangle\rangle$$

where \vec{a} is in W and \vec{z} is in $\perp @ W$. In fact, if $\{\vec{u}_1, \dots, \vec{u}_p\}$ is any orthogonal basis of W , then

$$\vec{a} = \sum_{i=1}^p \left(\frac{\vec{y} \cdot \vec{u}_i}{\vec{u}_i \cdot \vec{u}_i} \vec{u}_i \right) \quad \langle\langle 2 \rangle\rangle$$

and

$$\vec{z} := \vec{y} - \vec{a}$$

Proof:

Let $\{\vec{u}_1, \dots, \vec{u}_p\}$ be an orthogonal basis of W , and define \vec{a} as in the theorem. Clearly, \vec{a} is in W because \vec{a} is a linear combination of the basis $\vec{u}_1, \dots, \vec{u}_p$. Let $\vec{z} := \vec{y} - \vec{a}$

Since \vec{u}_1 is orthogonal to u_2, \dots, u_p , it follows from <<2>> that

$$\vec{z} \cdot \vec{u}_1 = (\vec{y} - \vec{a}) \cdot \vec{u}_1 = \vec{y} \cdot \vec{u}_1 - \vec{a} \cdot \vec{u}_1$$

substitute \vec{a} by its definition, and distribute, we have

$$\vec{y} \cdot \vec{u}_1 - \vec{a} \cdot \vec{u}_1 = \vec{y} \cdot \vec{u}_1 - \left(\frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} * \vec{u}_1 \right) \cdot \vec{u}_1 = \vec{y} \cdot \vec{u}_1 - \vec{y} \cdot \vec{u}_1 = 0$$

thus \vec{z} is orthogonal to \vec{u}_1 . Similarly, z is orthogonal to each u_j in the basis for W . Hence z is orthogonal to every vector in W . This is, z is in W^\perp .

To show that decomposition in <<1>> is unique, suppose that y can also be written as $y = a_1 + z_1$, with a_1 in W and z_1 in W^\perp . Then $a + z = a_1 + z_1$ (since both sides equal y), and so

$$a - a_1 = z_1 - z$$

This equality shows that the vector $v = a - a_1$ is in W and in W^\perp . (because z_1 and z are both in W^\perp , and W^\perp is a subspace)

Hence $v \cdot v = 0$, which shows that $v = \text{zero}$. This proves that $a = a_1$, and also $z = z_1$.

Property of orthogonal projection:

Let $W = \text{Span}\{u_1, \dots, u_p\}$. Let y be a vector in W . $\text{Projection}[y, W] = y$.

■ The best approximation theorem

Let W be a subspace of \mathbb{R}^n , y by any vector in \mathbb{R}^n , and v_a be the orthogonal projection of y onto W determined by an orthogonal basis of W . Then v_a is the closest point in W to y , in the sense that

$$\|y - v_a\| < \|y - v\| \quad \text{<<3>>}$$

for all v in W distinct from y .

Proof: Take v in W distinct from v_a . Then $v - v_a$ is in W . By the Orthogonal Decomposition Theorem, $y - v_a$ is orthogonal to W . In particular, $y - v_a$ is orthogonal to $v - v_a$. Since

$$y - v = (y - v_a) + (v_a - v)$$

The Pythagorean theorem gives

$$\|y - v\|^2 = \|y - v_a\|^2 + \|v_a - v\|^2$$

Now $\|v_a - v\|^2 > 0$ because $v_a - v \neq \text{zero}$, and so the inequality in <<3>> is clear.

If $\{u_1, \dots, u_p\}$ is an orthonormal basis for a subspace W of \mathbb{R}^n , then

$$\text{Projection}[y, W] = \sum_{i=1, \dots, p} (y \cdot u_i) * u_i \quad \text{<<4>>}$$

Let the matrix U has column vectors u_1, \dots, u_p . Then,

$$\text{Projection}[y, W] = U \cdot U^t \cdot y, \text{ for all } y \text{ in } \mathbb{R}^n. \quad \text{<<5>>}$$

Proof:

Formula <<4>> follows immediately from <<2>>. Also, <<4>> shows that $\text{projection}[y, W]$ is a linear combination of the columns of U using the weights $y \cdot u_i$. The weights may be written as $u_i^T \cdot y$, showing that they are the entries in $U^T \cdot y$ and justifying <<5>>.

7.4 The Gram-Schmidt process

7.4 The Gram-Schmidt process

Given a basis $\{x_1, \dots, x_p\}$ for a subspace W of \mathbb{R}^n , define

$$v_1 := x_1$$

$$v_p := x_p - \sum_{i=1}^{p-1} \frac{(x_p, v_i)}{(v_i, v_i)} v_i$$

Then $\{v_1, \dots, v_p\}$ is an orthogonal basis for W . In addition,

$$\text{Span}\{v_1, \dots, v_k\} = \text{Span}\{x_1, \dots, x_k\} \text{ for } 1 \leq k \leq p.$$

Proof:

For $1 \leq k \leq p$, let $W_k := \text{Span}\{x_1, \dots, x_k\}$. Set $v_1 := x_1$, so that $\text{Span}\{v_1\} = \text{Span}\{x_1\}$. Suppose that for some k we have constructed v_1, \dots, v_k so that $\{v_1, \dots, v_k\}$ is an orthogonal basis for W_k . Define

$$v_{(k+1)} := x_{(k+1)} - \text{Projection}[x_{(k+1)}, W_k]$$

Note that $\text{Projection}[x_{(k+1)}, W_k]$ is in W_k and hence also in $W_{(k+1)}$. Since $x_{(k+1)}$ is in $W_{(k+1)}$, so is $v_{(k+1)}$. Furthermore, $v_{(k+1)} \neq \text{zero}$ because $x_{(k+1)}$ is not in $W_k := \text{Span}\{x_1, \dots, x_k\}$. Hence $\{v_1, \dots, v_{(k+1)}\}$ is an orthogonal set of nonzero vectors in the $(k+1)$ -dimensional space $W_{(k+1)}$. By a theorem about number of independent vectors spans n -dimensional space, this set is an orthogonal basis for $W_{(k+1)}$. Hence $W_{(k+1)} = \text{Span}\{v_1, \dots, v_{(k+1)}\}$. After p steps, the process stops.

Idea:

The idea of Gram-Schmidt process is to use projection to go through each of the vectors. For each vector, project it onto the space previously found, thus obtaining an orthogonal basis one by one.

QR Factorization

If A is an $m \times n$ matrix with linearly independent columns, then A may be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for $\text{Col } A$ and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

Proof: The columns of A form a basis $\{x_1, \dots, x_n\}$ for $\text{Col } A$. Construct an orthonormal basis $\{v_1, \dots, v_n\}$ for $W = \text{Col } A$ with property $\langle v_i, v_j \rangle = \delta_{ij}$ in theorem 11. This basis may be constructed by the Gram-Schmidt process or some other means. Let

$$Q := [v_1 \ v_2 \ \dots \ v_n]$$

For $k=1, \dots, n$, x_k is in $\text{Span}\{x_1, \dots, x_k\} = \text{Span}\{v_1, \dots, v_k\}$. So there are constants, r_{1k}, \dots, r_{kk} , such that

$$x_k = r_{1k}v_1 + \dots + r_{kk}v_k + 0v_{(k+1)} + \dots + 0v_n$$

we may assume that $r_{kk} > 0$. (if $r_{kk} < 0$, multiply both r_{kk} and v_k by -1) This shows that x_k is a linear combination of the columns of Q using as weights the entries in the vector

$$r_k := [r_{1k}, \dots, r_{kk}, 0, \dots, 0]$$

That is, $x_k = Q r_k$ for $k=1, \dots, n$. Let $R := [r_1, \dots, r_n]$. Then

$$A = [x_1, \dots, x_n] = [Q r_1, \dots, Q r_n] = QR$$

The fact that R is invertible follows easily from the fact that the columns of A are linearly independent. Since R is clearly upper triangular, its nonnegative diagonal entries must be positive.

Questions

■ number of eigen values

What is the relation between a given square (real) matrix and the number of (real) eigen values it has? (I suppose if complex numbers and multiplicity are counted, then it's always the same as the dimension of the matrix?)

prove or study: The set of eigenvectors always spans row space?

Suppose A is a $n \times n$ matrix. I think that if we allow complex eigen values, then A always have n independent eigenvectors.

■ characteristic polynomial

What is the relation between characteristic polynomial and matrix without using determinants? i.e. I wish to see the connection of linear mapping and its characteristic polynomial, through aspects other than incomprehensible determinants or row reduction.

From: Robin Chapman <rjc@maths.ex.ac.uk>
 Newsgroups: sci.math
 Subject: Re: matrix and its polynomial
 Date: Wed, 24 Jun 1998 07:49:18 GMT
 Organization: University of Exeter

In article <yo33ecvuw5a.fsf@shell13.ba.best.com>,
 Xah Lee <xah@shell13.ba.best.com> wrote:
 > * How to define the characteristic polynomial of a square matrix
 > without involving determinants?

Let A be an n by n matrix over a field k . The characteristic polynomial of A is the product of $(X - a_j)^{n_j}$ where the a_j are the distinct eigenvalues of A over an algebraic closure of k , and n_j is the dimension of the nullspace of $(A - a_j I)^n$.

Robin Chapman + "They did not have proper
 Department of Mathematics - palms at home in Exeter."
 University of Exeter, EX4 4QE, UK +
 rjc@maths.exeter.ac.uk - Peter Carey,
 http://www.maths.ex.ac.uk/~rjc/rjc.html + Oscar and Lucinda

■ Relation between Null space and Column space

There is a theorem in basic linear algebra that says:

"Let A be an $m \times n$ matrix. Then the orthogonal complement of the row space of A is the nullspace of A ."

Is there a geometric insight that makes this apparant?

■ Transposition as a linear function

- Is there a square matrix B such that $B A = A^T$ for any square matrix A ?

Partial Answer: It seems yes.

To Do: Prove this, and find a general formula for B given A .

- Can we think of transposition of square matrices as one-to-one function for such matrices, and the fixed point of this function is all the identity matrices.

My Answer: Yes.

■ vector space always a sub space?

Is a vector space always a subspace of some vector space other than itself?

I think: A vector space may not be a subspace of some other vector space. For example: \mathbb{R}^3 is a vector space, but it is not a subspace of any other vector space, it is a subspace to itself only. Not True, because

$$\mathbb{R}^3 := \{ \{a,b,c\} \mid a, b, c \text{ are real numbers} \}.$$

We can defined another set

$$M := \{ \{a,b,c\} \mid a, b, c \text{ are complex numbers} \},$$

thus it can be shown that \mathbb{R}^3 proper subset of M , and since (i think) M is a vector space, so after all \mathbb{R}^3 is a subspace of some other vector space.

Now, is M a subspace of some other vector space?

In general, maybe we can always concoct a vector space V , so that the given vector space is a subset of V , thus a subspace of V .

To Do

Prove: Let A, M, N be matrices. Show that $A M + A N = A (M + N)$.

Coordinate permutation

■ Coordinate permutation

What kind of structural change occur to column vectors in a matrix during the process of row operation? Perhaps write a mma program illustrating this for 2D and 3D vectors.

1998/03/19.

Suppose we have a point with coordinates $\{a,b,c\}$. Now there are six permutations of its coordinate. If we connect all six points to each other, what kind of shape will it form? Especially consider a moving point. The problem can be generalized to higher dimensions.

I thought of it when studying linear algebra, on the paragraph that says "row operations does not change the linear dependence of the original's column vectors".

■ Solution

Some experiments with mma graphics shows that they are in general a six sided regular polygon lying on a plane, and the polygon has symmetry of that an equiangular triangle.

After thinking, here's my conclusion:

In the 2D analogous problem, the line $x=y$ acts like a mirror. Similarly, the 3D problem have 3 planes that are mirrors. Each plane is the symmetric plane between any two axes. If a point does not lie in one of these planes, then it will be reflected and results 6 points. The 6 points form a hexagon centered on the line $x=y=z$ and orthogonal to it, and it has symmetry of that an equiangular triangle. If we start with two or more points in 3D, then I imagine the result shape would be a prism-shaped in general, with both ends that of a hexagon. So it isn't very interesting. Before this, I thought the answer is some projection of higher dimension objects.

```
Needs["Graphics`Polyhedra`"]
Needs["GLExplorer`GLRenderer`"]
SetOptions[glMCreateWindow, Spin -> True, LockLights -> True];

GLShow[Graphics3D[{First@Polyhedron[Cube, {0, 0, 0}, 0.2],
  ({(First@Polyhedron[Tetrahedron, #, 0.05] &) /@#1, (Line[{{0, 0, 0}, #1}] &) /@#1} & )@
  Permutations@Table[If[Random[] > .5, 1, -1] * Random[], {3}]], Axes -> True,
  AxesLabel -> {"x", "y", "z"}, PlotRange -> {{-1, 1}, {-1, 1}, {-1, 1}}, Ticks -> None];
```

Geometry flavored questions

■ characteristic of sense-reversing mapping

Given $A.x$, how do we know that this mapping reverse sense?

A: probably by comparing a oriented triangle and its image.

■ characteristics of one-to-one mapping

Suppose we have a non-linear continuous mapping from R^n to R^m , with domain equal to R^n .

How can we tell whether this mapping is one-to-one?

■ geometric view of linear mapping that has maximum number of eigenvalues

Suppose A is an n by n square matrix.

Conjecture: A has maximum of n eigen values.

Conjecture: There exist an n by n matrix A with n eigen values, for any positive integer $n > 1$.

Question:

Suppose A is a matrix for a linear mapping from R^2 to R^2 , with two eigenvalues. I am unable to see intuitively the geometry aspect of such mapping.

If A is an n by n matrix with n eigen values. This means that the transformation has n lines that act like fixedpoints. Take $n=2$ case. For example, the matrix $\{\{3,-2\},\{1,0\}\}$ has two independent eigen vectors $\{\{1,1\},\{2,1\}\}$ and they are not orthogonal.

Answer: For the 2D case, view it as parallel projection of two planes in 3D. Similarly, higher dimension problem can be visualized using projection.

```
Needs["Transform2DPlot`"];

With[{matrix = {{3, -2}, {1, 0}}.{{Cos $\alpha$ , Cos[2 * Pi / 4 +  $\alpha$ ]}, {Sin[ $\alpha$ ], Sin[2 * Pi / 4 +  $\alpha$ ]}} /.
   $\alpha$  -> 2 * Pi / 4}, Transform2DPlot[matrix.{#1, #2} &, {-1, 1} 6, {-1, 1} 6, Prolog ->
  First@Transform2DPlot[{#1, #2} &, {-1, 1} 6, {-1, 1} 6, DisplayFunction -> Identity],
  Epilog -> {Hue[.7], Line /@ Transpose@(6 {-1 * #, #} &)@Eigenvectors@matrix},
  PlotRange -> {{-1, 1}, {-1, 1}} 7]];

With[{matrix = {{3, -2}, {1, 0}}.{{Cos $\alpha$ , Cos[2 * Pi / 4 +  $\alpha$ ]}, {Sin[ $\alpha$ ], Sin[2 * Pi / 4 +  $\alpha$ ]}} /.
   $\alpha$  -> -2 * Pi / 4}, (Transform2DGraphics[#, matrix.{#1, #2} &, Prolog -> #,
  Epilog -> {Hue[.7], Line /@ Transpose@(6 {-1 * #, #} &)@Eigenvectors@matrix},
  Axes -> True, AspectRatio -> Automatic, ResolutionLength -> 0.2] &@
  (PolarGrid[{0, 6, .1}, {1, 2, .1}, {Hue, GrayLevel[1 - #] &}])];
```

Typesetting Palette

```
NotebookPut@
(Notebook[#1, Active -> True, Background -> Hue[0.18, .5, 0.1], CellMargins -> {{3, 0}, {0, 3}},
  CellOpen -> True, Editable -> False, ImageMargins -> {{0, Inherited}, {Inherited, 0}},
  Magnification -> 1, PageWidth -> WindowWidth,
  ScrollingOptions -> {"PagewiseScrolling" -> True}, ShowCellBracket -> False,
  ShowCellLabel -> False, ShowCellTags -> False, WindowClickSelect -> False,
  WindowElements -> {}, WindowFrame -> "Normal", WindowFrameElements -> {"CloseBox"},
  WindowMargins -> {{Automatic, 30}, {Automatic, 5}},
  WindowSize -> {Fit, 410}, WindowTitle -> None, WindowToolbars -> {}] &@
((Cell[#1, Active -> True] &) /@ #1 &)@{BoxData@
  (GridBox[#1, RowSpacings -> 0, ColumnSpacings -> 0] &@ (Partition[#1, 5] &)@ (ButtonBox /@
    Join[({OverscriptBox[#1, "\u207a"] &}) /@ {"a", "b", "e", "u", "v", "x", "0", "\u25a1"}],
    {SubscriptBox[OverscriptBox["\u25a1", "\u207a"], "\u25a1"], "\u03bb", "B", "c", "\u2192", SuperscriptBox[
      "R", "n"], SuperscriptBox["R", "m"], SubscriptBox["\u25a1", "\u25a1"], SubscriptBox[
      RowBox[{"{"}, OverscriptBox["x", "\u207a"], "}"}, "B"}], Table[" ", {5}]]],
  BoxData@ (GridBox[#1, RowSpacings -> 0] &)@ ((List@ButtonBox@RowBox@List@
    RowBox@{SubscriptBox[#1, "1"], " ", "...", " ", SubscriptBox[#1, "n"]} &) /@
    {"a", "c", "x", "\u25a1"}), BoxData@ (GridBox[#1, RowSpacings -> 0] &)@
    ((List@ButtonBox@RowBox@{"{"}, RowBox[{SubscriptBox[OverscriptBox[#1, "\u207a"], "1"],
      " ", "...", " ", SubscriptBox[OverscriptBox[#1, "\u207a"], "n"}], "}"}) &) /@
    {"a", "b", "v", "\u25a1"}), BoxData@ (GridBox[#1, RowSpacings -> 0] &)@
  MapThread[List@ButtonBox@RowBox@List@RowBox@{SubscriptBox[#1, "1"],
    SubscriptBox[OverscriptBox[#2, "\u207a"], "1"], "+", "...", "+",
    SubscriptBox[#1, "n"], SubscriptBox[OverscriptBox[#2, "\u207a"], "n"]} &,
    Transpose@{{"x", "a"}, {"c", "v"}], BoxData@
  (GridBox[#1, RowSpacings -> 0] &)@ ((List@ButtonBox@# &) /@
    {RowBox[{SuperscriptBox["R", "n"], "\u2192", SuperscriptBox["R", "m"}]},
    RowBox@{RowBox[{"A", OverscriptBox["x", "\u207a"]}], "=", OverscriptBox["b", "\u207a"]},
    RowBox[OverscriptBox[u, "\u207a"], "+", OverscriptBox[v, "\u207a"]], RowBox[
      {UnderoverscriptBox["\u03a3", RowBox[{"i", "=", "1"}], "\u25a1"}, RowBox[{SubscriptBox[
        "\u25a1", "i"], " ", SubscriptBox[OverscriptBox["\u25a1", "\u207a"], "i"}]}]}]}];
```