About Pseudospherical Surfaces

If $X : M \subset \mathbf{R}^3$ is a surface with Gaussian curvature $K = -1$, then it is known that there exists a local asymptotic coordinate system (x, t) on M such that the first and second fundamental forms are

$$\mathrm{I} = dx^2 + dt^2 + 2\cos q \; dx\, dt, \quad \mathrm{II} = 2\sin q \; dx\, dt,$$

where q is the angle between asymptotic lines (the x-curves and t-curves). Such coordinates are called Tchebyshef coordinates. The Gauss-Codazzi equations for M in these coordinates become a single equation, the sine-Gordon equation (SGE)

$$q_{xt} = \sin q. \qquad (SGE)$$

Surfaces in $\mathbf{R}^3$ having constant Gauusian curvature K equal minus one are usually called *pseudospherical surfaces* (after the most well-known example, the pseudosphere) and the so-called Fundamental Theorem of Surfaces gives us a local correspondence between pseudospherical surfaces (up to rigid motion) and solutions of SGE. A general pseudospherical surface shares with the pseudosphere the fact that its intrinsic geometry is the hyperbolic geometry of Lobachevsky

Now classical results of Bäcklund and of Bianchi concerning pseudospherical surfaces provide methods to find many explicit solutions of the SGE and construct the corresponding pseudospherical surfaces. In fact, the SGE is one of the model soliton equations, and these classical method give rise to all of the soliton solutions of SGE. We will describe next a little of this very classical differential geometry.

Let M, M^* be surfaces in $\mathbf{R}^3$. A diffeomorphism $\ell : M \to M^*$ is called a *pseudospherical congruence* with constant θ if:

(i) the line joining p and $p^* = \ell(p)$ is tangent to both M and M^*,

(ii) the angle between the normal of M at p and the normal of M^* at $p^* = \ell(p)$ is θ, and

(iii) the distance from p to p^* is $\sin\theta$ for all $p \in M$.

The following result of Bäcklund is fundamental to the study of pseudospherical surfaces.

Bäcklund Theorem. *Let M, M^* be two surfaces in*

$\mathbf{R}^3$, *and* $\ell : M \to M^*$ *a pseudospherical congruence with constant* θ. *Then*

(a) *both* M *and* M^* *are pseudospherical surfaces,*

(b) *the Tchebyshef coordinates* x, t *on* M *maps to the Tchebyshef coordinates on* M^* *under* ℓ,

(c) *if* q *and* q^* *are the solutions of SGE corresponding to* M *and* M^* *respectively, then* q, q^* *satisfies*

$$\begin{cases} q^*_x = q_x + 4s\sin(\frac{q^*+q}{2}), \\ q^*_t = -q_t + \frac{2}{s}\sin(\frac{q^*-q}{2}), \end{cases} \qquad (\mathrm{BT}_\theta)$$

where $s = \tan\frac{\theta}{2}$.

Moreover, given q, system (BT_θ) is solvable for q^* if and only if q is a solution of the SGE, and the solution q^* is again a solution of the SGE.

We will call both ℓ and the transform from q to q^* a Bäcklund transformation. This description of Bäcklund transformations gives us an algorithm for generating families of solutions of the PDE by solving a pair of ordinary differential equations. The procedure can be repeated, but the miracle is that after the first step, the procedure can be carried out algebraically. This is the Bianchi Permutability Theorem.

Given two pseudospherical congruences $\ell_i : M_0 \to M_i$ with angles θ_i respectively and $\sin\theta_1^2 \neq \sin\theta_2^2$, then there exist an algebraic construction of a unique surface M_3, and pseudospherical congruences $\tilde{\ell}_1 : M_2 \to M_3$ and $\tilde{\ell}_2 : M_1 \to M_3$ with angles θ_1 and θ_2 respectively such that $\tilde{\ell}_2\ell_1 = \tilde{\ell}_1\ell_2$. The analytic reformulation of this theorem is the following: Suppose q is a solution of the SGE and q_1, q_2 are two solutions of system (BT_θ) with angles $\theta = \theta_1, \theta_2$ respectively. The Bianchi permutability theorem gives a third local solution q_3 to the SGE

$$\tan\frac{q_3 - q}{4} = \frac{s_1 + s_2}{s_1 - s_2}\tan\frac{q_1 - q_2}{4},$$

where $s_1 = \tan\frac{\theta_i}{2}$ and $s_2 = \tan\frac{\theta_2}{2}$.

To see how the scheme works, we start with the trivial solution $q = 0$ of SGE, then (BT_θ) can be solved explicitly to get

$$q^*(x,t) = 4\tan^{-1}(e^{sx+\frac{t}{s}}), \tag{1}$$

the 1-soliton solutions of SGE. (Here $s = \tan\frac{\theta}{2}$.) An application of the Permutability theorem then give the 2-soliton solutions

$$q(x,t) = 4\tan^{-1}\left(\frac{s_1+s_2}{s_1-s_2}\,\frac{e^{s_1x+\frac{1}{s_1}t}-e^{s_2x+\frac{1}{s_2}t}}{1+e^{(s_1+s_2)x+(\frac{1}{s_1}+\frac{1}{s_2})t}}\right). \tag{2}$$

Repeated applications of the Permutability theorem give complicated but nevertheless explicit n-soliton solutions. Note that the parameters s_1, s_2 in the above formula for 2-solitons are real. But for $s_1 = e^{i\theta}$ and $s_2 = -e^{-i\theta}$, even though q_1, q_2 are not real-valued, nevertheless

$$q_3(x,t) = 4\tan^{-1}\left(\frac{\sin\theta\sin(T\cos\theta)}{\cos\theta\cosh(X\sin\theta)}\right) \tag{3}$$

is real and a solution of SGE, where $X = x-t$, and $T = x+t$ are space-time coordinates. This solution is periodic in T and is called a *Breather*.

The "surface" corresponding to $q = 0$ is degenerate and in fact is a straight line. The surfaces corresponding to

(i) 1-soliton (formula (1)) with $s = 1$ is the Pseudosphere,

(ii) 1-soliton (formula (1)) with $s \neq 1$ is a Dini Surface,

(iii) 2-soliton (formula (2)) contains the Kuen Surface.

(iv) Breather solution (formula (3)) with $\cos\theta$ a rational number is a pseudospherical surface periodic in the T direction.

Even though the breather solution q is periodic in T, the corresponding pseudospherical surface may not be periodic in T. This is because when we use the Fundamental Theorem of Surfaces to construct the surface from solution q of the SGE, we need to solve two compatible ODEs whose coefficients are given by functions of q and q_x. For Breathers, the solutions of these ODEs are periodic in T if $\cos\theta$ is rational.

You will notice that all of the pseudospherical surfaces shown in the program have obvious singularities. In fact, a theorem of Hilbert says that the hyperbolic plane can not be isometrically immersed in $\mathbf{R}^3$, and this implies that all complete pseudospherical surfaces must have singularities. Although soliton solutions are smooth on the entire (x, t)-plane, the corresponding pseudospherical surfaces have singularities where the induced metric becomes degenerate, i.e., where

$$\det \begin{pmatrix} 1 & \cos q \\ \cos q & 1 \end{pmatrix} = 0.$$

In other words, a surface corresponding to a global solution q of SGE will have a singularities along the curves where q is a multiple of π. Moreover, since the metric has rank 1 there, these surfaces have cusp singularities.

For an elementary and short introduction to soliton theory and its relation to SGE, see the article by C. L. Terng and K. Uhlenbeck: Geometry of Solitons, Notices of AMS, 47(2000), 17-25. One may also download the pdf file of this paper from

http://www.math.neu.edu/~terng/MyPapers.html

CLT