LECTURE NOTES ON CURVES AND SURFACES IN $\mathbb{R}^3$

CHUU-LIAN TERNG[1]

Preliminary version and in progress, April 2, 2003

Contents

1. **Introduction** 1
2. **Curves in $\mathbb{R}^2$ and in $\mathbb{R}^3$** 2
2.1. **Some results from ODE** 3
2.2. **Curves in $\mathbb{R}^2$** 5
2.3. **Space curves** 9
2.4. **Frobenius Theorem** 13
2.5. **Smoke-ring equation and the non-linear Schrödinger equation** 15
3. **Surfaces in $\mathbb{R}^3$** 21
3.1. **Some linear algebra** 21
3.2. **Embedded surfaces in $\mathbb{R}^3$** 25
3.3. **The first and second fundamental forms** 30
3.4. **The Gaussian curvature and mean curvature** 34
4. **Calculus of variations** 38
4.1. **Calculus of Variations of one variable** 39
4.2. **Geodesics** 41
4.3. **Calculus of variations of two variables** 44
4.4. **Minimal surfaces** 46
5. **Fundamental Theorem of surfaces in $\mathbb{R}^3$** 47
5.1. **Gauss-Codazzi equations in arbitrary local coordinates** 48
5.2. **The Gauss Theorem** 52
5.3. **Gauss-Codazzi equation in orthogonal coordinates** 55
5.4. **Line of curvature coordinates** 57
6. **Surfaces in $\mathbb{R}^3$ with $K = -1$** 58
6.1. **$K = -1$ surfaces in $\mathbb{R}^3$ and the sine-Gordon equation** 59
6.2. **Moving frame method** 62
6.3. **Bäcklund Theorem** 69

[1]Research supported in part by NSF Grant NSF DMS-9972172 and NCTS of Taiwan.

1. Introduction

To study a collection of geometric objects, we often proceed as follows:

(1) Specify a group of "symmetries" that acts on objects, and define two objects a, b to be equivalent if and only if $b = g(a)$ for some g in the symmetry group,
(2) Find a complete set of invariants and their relations. This means that we want to associate to each object a a set of quantities $I(a)$ satisfying certain relations so that
   (i) if a and b are equivalent, then $I(a) = I(b)$,
   (ii) given a set $\mathcal{I}$ satisfies the relations then there exists an object a unique up to equivalence so that $I(a) = \mathcal{I}$.

One of the simplest examples is the study of triangles in $\mathbb{R}^3$ with the group of rigid motions as symmetry group. Recall that a rigid motion is a rotation follows by a translation. It follows from plane geometry that the length of three sides $I(a) = \{s_1, s_2, s_3\}$ of a triangle a forms a complete set of invariants and they satisfy the relations (triangle inequalities and length is positive):

$$s_1 + s_2 > s_3, \quad s_2 + s_3 > s_1, \quad s_1 + s_3 > s_2, \quad s_i > 0, \quad 1 \le i \le 3.$$

In this series of lectures, we use differential calculus, linear algebra, and differential equations to study curves and surfaces in $\mathbb{R}^3$ with the rigid motions as symmetry group. For example, for

- the study of curves in $\mathbb{R}^2$, the objects are immersed curves in $\mathbb{R}^2$, and the curvature function of a curve is a complete set of invariants,
- the study of curves in $\mathbb{R}^3$, the objects are immersed curves in $\mathbb{R}^3$, and the curvature k and torsion τ of a curve form a complete set of invariants with condition $k > 0$,
- the study of surfaces in $\mathbb{R}^3$, the objects are immersed surfaces in $\mathbb{R}^3$, and their first and second fundamental forms form a complete set of local invariants and they must satisfy the Gauss-Codazzi equations.

These invariants determine the local geometry of curves and surfaces unique up to rigid motions. One goal of these lectures is to explain the Fundamental theorems of curves and surfaces in $\mathbb{R}^3$. Another goal is to demonstrate by examples how to construct soliton equations from natural geometric curve flows and from the Gauss-Codazzi equations for surfaces whose numerical invariants satisfy certain natural conditions. The examples we will give are:

(1) the evolution of local invariants of the evolution of space curves move in the direction of the binormal with curvature as its speed is the non-linear Schrödinger equation,
(2) the Gauss-Codazzi equation for surfaces in $\mathbb{R}^3$ with $K = -1$ is the sine-Gordon equation,
(3) the Gauss-Codazzi equation for surfaces in $\mathbb{R}^3$ admitting a conformal line of curvature coordinate system is a reduced 3-wave equation.

2. Curves in $\mathbb{R}^2$ and in $\mathbb{R}^3$

Let $\cdot$ denote the standard dot product in $\mathbb{R}^n$, and $\parallel v \parallel = \sqrt{v \cdot v}$ the length of $v \in \mathbb{R}^n$. In this section, a curve in $\mathbb{R}^n$ means a smooth map $\alpha : [a, b] \to \mathbb{R}^n$ so that $\alpha'(t) \neq 0$ for all $t \in [a, b]$, i.e., α is an *immersion*. The arc length of α from $t_0 \in [a, b]$ to t is

$$s(t) = \int_{t_0}^{t} \parallel \alpha'(t) \parallel \, dt.$$

Since $\frac{ds}{dt} = \parallel \alpha' \parallel$ and $\alpha'(t) \neq 0$ for all $t \in [a, b]$, the function $s : [a, b] \to [0, \ell]$ is a diffeomorphism, where ℓ is the arc length of α. So we can make a change of coordinate $t = t(s)$ (the inverse function of $s = s(t)$). Moreover, by the chain rule we have

$$\frac{d\alpha}{ds} = \frac{\frac{d\alpha}{dt}}{\frac{dt}{ds}} = \frac{\alpha'(t)}{\parallel \alpha'(t) \parallel},$$

i.e., $\frac{d\alpha}{ds}$ is a unit vector tangent to α.

A curve $\alpha : [c_1, c_2] \to \mathbb{R}^n$ is said to be *parametrized by its arc length* if $\parallel \alpha'(t) \parallel = 1$ for all $t \in [c_1, c_2]$. The above discussion says that any curve in $\mathbb{R}^n$ can be parametrized by its arc length.

2.1. Some results from ODE.

Proposition 2.1.1. *Suppose $e_1, \cdots, e_n : [a, b] \to \mathbb{R}^n$ are smooth maps, and $\{e_1(t), \cdots, e_n(t)\}$ is an orthonormal (o.n.) basis of $\mathbb{R}^n$ for all $t \in [a, b]$. Then*

(1) *there exists a matrix valued function $A(t) = (a_{ij}(t))$ so that*

$$e_i'(t) = \sum_{j=1}^{n} a_{ji}(t) e_j(t), \tag{2.1.1}$$

(2) $a_{ij} = e_j' \cdot e_i$.

(3) $a_{ij} + a_{ji} = 0$, *i.e.*, $A(t)$ *is skew-symmetric*

Proof. Since $e_1, \cdots, e_n$ form an o.n. basis of $\mathbb{R}^n$, $e_i'(t)$ can be written as a linear combination of $e_1(t), \cdots, e_n(t)$. This proves (1). Dot (2.1.1) with e_j to get (2). Because $e_i \cdot e_j = \delta_{ij}$, the derivative of this equation implies that $e_i' \cdot e_j + e_i \cdot e_j' = 0$, which proves (3). □

It follows from the definition of matrix multiplication that

$$(v_1, v_2, \cdots, v_n) \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_m \end{pmatrix} = \sum_k x_k v_k,$$

where $v_1, \cdots, v_n$ are column vectors. So we can rewrite (2.1.1) in matrix form

$$g'(t) = g(t)A(t), \quad \text{where } g(t) = (e_1(t), \cdots, e_n(t)), \quad A(t) = (a_{ij}(t)).$$

Note the i-th column of A gives the coefficient of of e_i' with respect to the o.n. basis $e_1, \cdots, e_n$.

The following is a special case of the existence and uniqueness theorem for solutions of ODE:

Theorem 2.1.2. (Existence and Uniqueness of ODE) *Let I be an interval of $\mathbb{R}$, U an open subset of $\mathbb{R}^n$, and $F : I \times U \to \mathbb{R}^n$ a smooth map. Then given $t_0 \in I$ and $p_0 \in U$, there exists $\delta > 0$ and a unique smooth $x : (t_0 - \delta, t_0 + \delta) \to U$ so that x is a solution of the following initial value problem:*

$$\begin{cases} x'(t) = F(t, x(t)), \\ x(t_0) = p_0 \end{cases} \tag{2.1.2}$$

Let $gl(n)$ denote the space of all $n \times n$ real matrices. An $n \times n$ matrix C is called *orthogonal* if $C^tC = \mathrm{I}$. It follows from definition of matrix product that we have

Proposition 2.1.3. *The following statements are equivalent for a matrix $C \in gl(n)$:*

(1) *C is orthogonal,*
(2) *the column vectors of C are o.n.,*
(3) *the row vectors of C are o.n..*

Proposition 2.1.4. *Suppose $A(t) = (a_{ij}(t))$ is skew symmetric for all t. If $C \in gl(n)$ is an orthogonal matrix, $t_0 \in (a, b)$, and $g : (a, b) \to gl(n)$ is the solution of*

$$g'(t) = g(t)A(t), \quad g(t_0) = C,$$

then $g(t)$ is orthogonal for all $t \in (a, b)$.

Proof. Set $y = g^t g$. We compute the derivative y' to get

$$y' = (g^t)'g + g^t g' = (g')^t g + g^t g' = (gA)^t g + g^t(gA) = A^t y + yA.$$

Since the constant function $y(t) = I$ is a solution with initial data $y(t_0) = \mathrm{I}$, it follows from the uniquness of solutions of ODE (Theorem 2.1.2) that $y(t) = \mathrm{I}$ for all $t \in (a, b)$. □

Exercise 2.1.5.

(1) For $X, Y \in gl(n)$, let $(X, Y) = \mathrm{tr}(X^t Y)$. Show that $(\, ,)$ is an inner product on $gl(n)$, and $gl(n)$ is isometric to the Euclidean space $\mathbb{R}^{n^2}$ with the standard inner product.

(2) Given $A \in gl(n)$, prove that

$$y(t) = \sum_{j=0}^{\infty} \frac{A^n}{n!} \, t^n$$

converges uniformly on any finite interval $[-r, r]$. Hence $y(t)$ is continuous. Prove also that y is differentiable and y is the solution of the initial value problem:

$$y'(t) = Ay(t), \quad y(0) = \mathrm{Id}.$$

We will denote $y(t)$ as e^{tA}.

(3) Compute e^{tA} for $A = \mathrm{diag}(a_1, \ldots, a_n)$ and for $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(4) Suppose D is the diagonal matrix $\mathrm{diag}(d_1, \cdots, d_n)$, g is an invertible $n \times n$ matrix, and $A = gDg^{-1}$. Show that $e^{tA} = ge^{tD}g^{-1}$.

2.2. Curves in $\mathbb{R}^2$.

Let $\alpha(t) = (x(t), y(t))$ be a smooth curve parametrized by its arc length. Then $e_1(t) = \alpha'(t) = (x'(t), y'(t))$ is a unit vector tangent to α at $\alpha(t)$. Note that $(-x_2, x_1)$ is perpendicular to (x_1, x_2). Let $e_2(t) = (-y'(t), x'(t))$. Then $e_1(t), e_2(t)$ form an o.n. basis of $\mathbb{R}^2$. Such $(e_1(t), e_2(t))$ is called a *o.n. moving frame* along α. Since a 2×2 skew-symmetric matrix is of the form

$$\begin{pmatrix} 0 & -u \\ u & 0 \end{pmatrix}.$$

It follows from Proposition 2.1.1 that there exists a smooth function $k(t)$ so that

$$(e_1, e_2)' = (e_1, e_2) \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}.$$

In fact, we have

$$k = e_1' \cdot e_2,$$

and

$$\begin{cases} e_1' = ke_2, \\ e_2' = -ke_1, \end{cases}$$

The function k is called the *curvature* of the curve α.

Example 2.2.1. A straight line can be parametrized as $\alpha(t) = p_0 + tu$, where $p_0 \in \mathbb{R}^2$ and u a constant unit vector in $\mathbb{R}^2$. Then $e_1 = u$ a constant, and $e_1' = 0$. Hence the curvature of a straight line is identically 0. Conversely, if the curvature of a curve β parametrized by arc length is identially zero, then β must be a straight line. To see this, note first that $e_1' = \beta'' = ke_2 = 0$. Hence e_1 is a constant vector, say u. But $(\beta(t) - tu)' = e_1 - u = 0$ implies that $\beta - tu$ is a constant p_0. So $\beta(t) = p_0 + tu$ is a straight line. So the curvature of a plane curve measures the deviation of the curve being a straight line.

Example 2.2.2. The circle of radius r centered at p_0 is

$$\alpha(t) = p_0 + \left(r \cos \frac{t}{r}, r \sin \frac{t}{r} \right).$$

Since $\alpha'(t) = \left(-\sin \frac{t}{r}, \cos \frac{t}{r}\right)$ has length 1, t is an arc length parameter. A direct computation shows that the curvature is the constant function $\frac{1}{r}$.

The following Proposition says that the curvature is the instantanous rate of change of the polar angle of tangent of the curve. So when k is large, the curve winds around faster.

Proposition 2.2.3. *Suppose $\alpha(t) = (x(t), y(t))$ is parametrized by its arc length, and $\theta(t)$ the angle from $(1,0)$ to $\alpha'(t)$. Then $\theta'(t) = k(t)$.*

Proof. Since $e_1 = \alpha' = (\cos\theta(t), \sin\theta(t))$, $e_2(t) = (-\sin\theta(t), \cos\theta(t))$. A direct computation implies that $k = e_1' \cdot e_2 = \theta'(t)$. □

Note that the tangent line to α at $\alpha(t_0)$ gives the best linear approximation to α near $\alpha(t_0)$. The best quadratic approximation of α is the osculating circle. We explain this next.

Definition 2.2.4. The osculating circle at $\alpha(t_0)$ of a curve α is the circle of radius $r = \frac{1}{k(t_0)}$ centered at $\alpha(t_0) + re_2(t_0)$.

Proposition 2.2.5. *Suppose $\alpha(t) = (x(t), y(t))$ is a plane curve, and t is the arc length parameter. Then the osculating circle C at $\alpha(t_0)$ is the best second order approximation of α near t_0.*

Proof. We may assume that $t_0 = 0$, $\alpha(0) = (0,0)$, $\alpha'(0) = (1,0)$, and $e_2(0) = (0,1)$. Let $r = \frac{1}{k(0)}$. Since $e_1' = ke_2$, $\alpha''(0) = \frac{1}{r}(0,1)$. So the degree 2 Taylor series of $\alpha(t) = (x(t), y(t))$ at $t = 0$ is

$$(x(0) + x'(0)t + \frac{1}{2}x''(0)t^2, y(0) + y'(0)t + \frac{1}{2}y''(0)t^2) = \left(t, \frac{t^2}{2r}\right).$$

The osculating circle at $\alpha(0) = (0,0)$ is the circle of radius r centered at $(0,0) + r(0,1) = (0,r)$. So

$$C(t) = (0,r) + r\left(\sin\frac{t}{r}, -\cos\frac{t}{r}\right)$$

gives an arc length parameter of the osculating circle at $\alpha(0)$. It is easy to check that the degree two Taylor series of $C(t)$ at $t = 0$ is the same as the one for $\alpha(t)$. □

Theorem 2.2.6. (Fundamental Theorem of Curves in $\mathbb{R}^2$)

(1) *Given a smooth function $k : (a,b) \to \mathbb{R}$, $p_0 \in \mathbb{R}^2$, $t_0 \in (a,b)$, and u_1, u_2 a fixed o.n. basis of $\mathbb{R}^2$, there exists a unique curve $\alpha(t)$ defined in a small neighborhood of t_0 so that t is an arc length parameter, $\alpha(0) = p_0$, $\alpha'(0) = u_1$, and $k(t)$ is its curvature function.*

(2) *Let $\alpha, \tilde{\alpha} : (-\delta, \delta) \to \mathbb{R}^2$ be smooth curves parametrized by arc length. If α and $\tilde{\alpha}$ have the same curvature function, then there exists a rigid motion T so that $\tilde{\alpha}(t) = T(\alpha(t))$ for all t.*

Proof. (1) To construct the curve α, we need to solve

$$\begin{cases} \alpha' = e_1, \\ e_1' = ke_2, \\ e_2' = -ke_1 \end{cases} \tag{2.2.1}$$

with initial data $\alpha(0) = p_0$, $e_1(0) = u_1$, $e_2(0) = u_2$. Let $F : (a, b) \times (\mathbb{R}^2)^3 \to (\mathbb{R}^2)^3$ be the smooth map defined by

$$F(t, y_0, y_1, y_2) = (y_1, k(t)y_2, -k(t)y_3).$$

It follows from Theorem 2.1.2 that there exists a unique solution with initial data $(y_0, y_1, y_2)(0) = (p_0, u_1, u_2)$. Since $\begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}$ is skew-symmetric and

$$(y_1, y_2)' = (y_1, y_2) \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix},$$

it follows from Proposition 2.1.4 that $y_1(t), y_2(t)$ form an o.n. basis for each t.

(2) Let $e_1 = \alpha'$, e_2 the unit normal for α so that $\det(e_1, e_2) = 1$, and $\tilde{e}_1 = \tilde{\alpha}'$, $\tilde{e}_2$ the unit normal for $\tilde{\alpha}$ so that $\det(\tilde{e}_1, \tilde{e}_2) = 1$. By assumption α and $\tilde{\alpha}$ have the same curvature function k, so both (α, e_1, e_2) and $(\tilde{\alpha}, \tilde{e}_1, \tilde{e}_2)$ satisfy the same differential equation

$$y_0' = y_1, \quad y_1' = ky_2, \quad y_2' = -ky_1. \tag{2.2.2}$$

Let h denote the constant rotation

$$h = (\tilde{e}_1(0), \tilde{e}_2(0))(e_1(0), e_2(0))^{-1},$$

and T the rigid motion

$$T(v) = \tilde{\alpha}(0) - h(\alpha(0)) + hv.$$

Set $\beta(t) = T(\alpha(t))$. Then $\beta(0) = \tilde{\alpha}(0)$ and $\beta' = he_1$. Since h is a rotation, he_2 is the unit normal of β and (he_1, he_2) is a local o.n. frame along β. By definition of h, we have $h(e_i(0)) = \tilde{e}_i(0)$ for $i = 1, 2$. But

$$\begin{cases} \beta' = he_1, \\ (he_1)' = h(e_1') = h(ke_2) = kh(e_2), \\ (h(e_2)' = h(e_2') = h(-ke_1) = -kh(e_1). \end{cases}$$

So both $\tilde{a}$ and β are solutions of the following initial value problem:

$$\begin{cases} y_0' = y_1, \\ y_1' = ky_2, \\ y_2' = -ky_1, \\ (y_0, y_1, y_2)(0) = (\tilde{\alpha}(0), \tilde{e}_1(0), \tilde{e}_2(0)). \end{cases}$$

By the uniqueness of ODE, $\beta = \tilde{\alpha}$, i.e., $\tilde{\alpha} = T(\alpha)$. □

Let $\alpha : [0, \ell] \to \mathbb{R}^2$ be a smooth curve parametrized by its arc length, and $e_1 = \alpha'$ and e_2 the unit normal. Below we give three classical ways to generate new curves from α:

(1) The *involute* of α starting at $\alpha(t_0)$ is the curve
$$\beta_{t_0}(t) = \alpha(t + t_0) - t e_1(t + t_0).$$
Note that if α is a simple close curve, then the involute of α at $\alpha(t_0)$ can be viewed as pulling the thread from a spool of the shape of α from the point $\alpha(t_0)$.

(2) The *evolute* of a curve α is the locus of the centers of osculating circles of α, i.e., the evolute is
$$\beta(t) = \alpha(t) + \frac{1}{k(t)} e_2(t).$$

(3) The curve *parallel* to α with distance r is
$$\alpha(t) + r e_2(t).$$

Exercise 2.2.7.

(1) Recall that a map $T : \mathbb{R}^n \to \mathbb{R}^n$ is a *rigid motion* if T is of the form $T(x) = gx + b$ for some $n \times n$ orthogonal matrix g and $b \in \mathbb{R}^n$. Prove that the set of all rigid motions of $\mathbb{R}^n$ with composition as multiplication is a group.
(2) Let α be a curve in $\mathbb{R}^2$ parametrized by arc length, k its curvature, and T a rigid motion. Prove that the curvature of $\beta = T \circ \alpha$ is also k.
(3) Suppose $\alpha(t)$ is not parametrized by arc length, write down the curvature function in t variable.
(4) Prove that the curve parallel to α with distance r fails to be an immersed curve at t_0 if and only if the radius of osculating circle at $\alpha(t_0)$ is $\frac{1}{r}$. Use 3D-XplorMath to visualize this.
(5) Use 3D-XplorMath Plane curve category to see the involutes of a curve seem to be all parallel. Use mathematics to either prove or disprove what you see.
(6) Write down the curve obtained by tracing a fixed point on a circle of radius a when the circle is rolling along the x-axis with a fixed speed b. This curve is called a *cycloid*.
(7) Show that the evolute of the cycloid is a translation of the same cycloid.

2.3. **Space curves.**

Let $\alpha(t)$ be a curve in $\mathbb{R}^3$ parametrized by arc length. As for plane curves, we need to construct a moving o.n. frame $e_1(t), e_2(t), e_3(t)$ along α such that $e_1(t) = \alpha'(t)$. Since the normal space at every point is two dimension, it is not clear which normal direction should be e_2. There are two natural ways to choose e_2. The first is the classical Frenet frame. Since $e_1 = \alpha'$ has unit length,

$$(e_1 \cdot e_1)' = 2e_1' \cdot e_1 = 0.$$

So $\alpha'' = e_1'$ is perpendicular to e_1. Assume that $\alpha''(t)$ never vanishes in an interval (a, b). Let $e_2(t)$ denote the unit direction along $\alpha''(t)$ and $k(t)$ the length of $\alpha''(t)$, i.e., $e_1' = ke_2$. Let $e_3 = e_1 \times e_2$. So $\{e_1, e_2, e_3\}$ is an o.n. basis of $\mathbb{R}^3$ for all t. By Proposition 2.1.1, there exists a skew symmetric 3×3 matrix valued map $A(t)$ so that

$$(e_1, e_2, e_3)' = (e_1, e_2, e_3)A.$$

Since 3×3 skew-symmetric matrix must be of the form

$$\begin{pmatrix} 0 & -f_1 & -f_2 \\ f_1 & 0 & -f_3 \\ f_2 & f_3 & 0 \end{pmatrix}$$

and $e_1' = ke_2$, the first column of A must be $(0, k, 0)^t$ and there exists τ so that

$$(e_1, e_2, e_3)' = (e_1, e_2, e_3) \begin{pmatrix} 0 & -k & 0 \\ k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}. \tag{2.3.1}$$

Note that $\tau = e_2' \cdot e_3$. The functions k and τ are called the *curvature* and *torsion* respectively, e_2 the *normal*, e_3 the *binormal*, and (e_1, e_2, e_3) is the *Frenet frame* along α. Equation (2.3.1) is called the *Frenet-Serret equation*, which can also be written as

$$\begin{cases} e_1' = ke_2, \\ e_2' = -ke_1 + \tau e_2, \\ e_3' = -\tau e_2. \end{cases} \tag{2.3.2}$$

Proposition 2.3.1. *If the torsion of α is identically zero, then α must lie in a plane.*

Proof. Since $\tau = 0$, the third equation of (2.3.2) implies that $e_3' = -\tau e_2 = 0$. Hence $e_3 = b$ is a constant vector. The derivative of $\alpha(t) \cdot b$ is $e_1 \cdot b = e_1 \cdot e_3 = 0$. So $\alpha(t) \cdot b = c_0$ a constant. □

So torsion of a space curve measures the deviation of α being a plane curve.

Theorem 2.3.2. (Fundamental Theorem of curves in $\mathbb{R}^3$)

(1) *Given smooth functions $k, \tau : (a,b) \to \mathbb{R}$ so that $k(t) > 0$, $t_0 \in (a,b)$, $p_0 \in \mathbb{R}^3$ and (u_1, u_2, u_3) a fixed o.n. basis of $\mathbb{R}^3$, there exists $\delta > 0$ and a unique curve $\alpha : (t_0 - \delta, t_0 + \delta) \to \mathbb{R}^3$ parametrized by arc length such that $\alpha(0) = p_0$, and (u_1, u_2, u_3) is the Frenet frame of α at $t = t_0$.*

(2) *Suppose $\alpha, \tilde{\alpha} : (a,b) \to \mathbb{R}^3$ are curves parametrized by arc length, and $\alpha, \tilde{\alpha}$ have the same curvature function k and torsion function τ. Then there exists a rigid motion T so that $\tilde{\alpha} = T(\alpha)$.*

Proof. This theorem can be proved in exactly the same way as Theorem 2.2.6 by applying the existence and uniqueness Theorem 2.1.2 of solution of ODE to $F : (a,b) \times (\mathbb{R}^3)^3 \to (\mathbb{R}^3)^3$ defined by

$$F(t, y_0, y_1, y_2) = (y_1, k(t)y_2, -k(t)y_1 + \tau(t)y_3, -\tau(t)y_2).$$

□

Next we describe parallel frames. Let (e_1, ξ_1, ξ_2) be a smooth o.n. frame along α such that $e_1 = \alpha'$. Then by Proposition 2.1.1, there exist three smooth functions f_1, f_2, f_3 so that

$$(e_1, \xi_1, \xi_2)' = (e_1, \xi_1, \xi_2) \begin{pmatrix} 0 & -f_1 & -f_2 \\ f_1 & 0 & -f_3 \\ f_2 & f_3 & 0 \end{pmatrix}.$$

We want to change the o.n. normal frame (ξ_1, ξ_2) to (v_1, v_2) so that the 23 entry of the coefficient matrix of $(e_1, v_1, v_2)'$ is zero. To do this, we rotate the normal frame $\xi_1(t), \xi_2(t)$ by an angle $\beta(t)$:

$$\begin{cases} v_1(t) = \cos\beta(t)\xi_1 + \sin\beta(t)\xi_2, \\ v_2(t) = -\sin\beta(t)\xi_1 + \cos\beta(t)\xi_2. \end{cases}$$

We want to choose β so that $v_1' \cdot v_2 = 0$. But

$$\begin{aligned} v_1' \cdot v_2 &= \beta'(-\sin\beta\ \xi_1 + \cos\beta\ \xi_2) \cdot (-\sin\beta\ \xi_1 + \cos\beta\ \xi_2) \\ &\quad + (\cos\beta\ \xi_1' + \sin\beta\ \xi_2') \cdot (-\sin\beta\ \xi_1 + \cos\beta\ \xi_2) \\ &= \beta' + (\cos^2\beta\ \xi_1' \cdot \xi_2 - \sin^2\beta\ \xi_2' \cdot \xi_1) \\ &= \beta' + f_3. \end{aligned}$$

So if we choose $\beta(t)$ so that

$$\beta' = -f_3, \tag{2.3.3}$$

then the new o.n. frame (e_1, v_1, v_2) satisfies

$$(e_1, v_1, v_2)' = (e_1, v_1, v_2) \begin{pmatrix} 0 & -k_1 & -k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{pmatrix}, \tag{2.3.4}$$

where
(2.3.5)

$$\begin{cases} k_1 = e_1' \cdot v_1 = e_1' \cdot (\cos\beta\ \xi_1 + \sin\beta\ \xi_2) = f_1 \cos\beta + f_2 \sin\beta, \\ k_2 = e_1' \cdot v_2 = -f_1 \sin\beta + f_2 \cos\beta. \end{cases}$$

The o.n. frame (e_1, v_1, v_2) is called a *parallel frame* along α. The reason for the name parallel is because the normal component of the derivative of the normal vector field v_i is zero. The functions k_1, k_2 are called the *prinicipal curvatures* along v_1, v_2 respectively. However, the choice of parallel frame is not unique because β only need to satisfy (2.3.3), so we can replace β by β plus a constant θ_0. If we rotate (v_1, v_2) by a constant rotation θ_0 to $(\tilde{v}_1, \tilde{v}_2)$, then $(e_1, \tilde{v}_1, \tilde{v}_2)$ is again parallel. But the principal curvature $\tilde{k}_1$ along $\tilde{v}_1$ is

$$\tilde{k}_1 = e_1' \cdot \tilde{v}_1 = e_1' \cdot (\cos\theta_0 v_1 + \sin\theta_0 v_2) = \cos\theta_0 k_1 + \sin\theta_0 k_2.$$

Similar computation gives

$$\tilde{k}_2 = -\sin\theta_0 k_1 + \cos\theta_0 k_2.$$

Note that

$$\alpha'' = e_1' = k_1 v_1 + k_2 v_2 = \tilde{k}_1 \tilde{v}_1 + \tilde{k}_2 \tilde{v}_2.$$

So we have

Proposition 2.3.3. *Let α be a curve in $\mathbb{R}^3$ parametrized by arc length, (v_1, v_2) a parallel normal frame along α, and k_1, k_2 principal curvature with respect to v_1, v_2 respectively. Then $\alpha'' = k_1 v_1 + k_2 v_2$.*

Corollary 2.3.4. *Let (v_1, v_2) be a parallel normal frame for α, k_1, k_2 the corresponding principal curvatures, (e_1, e_2, e_3) the Frenet frame, and k, τ the curvature and torsion of α. Then*

(1) *there exists $\beta(t)$ so that $v_1 = \cos\beta\ e_2 + \sin\beta\ e_3$, $v_2 = -\sin\beta\ e_2 + \cos\beta\ e_3$, and $\beta' = -\tau$,*
(2) *$k_1 = k\cos\beta$, $k_2 = -k\sin\beta$.*

Proof. Use $\xi_1 = e_2, \xi_2 = e_3$ in the computation before Proposition 2.3.3. Since $f_1 = k$, $f_2 = 0$ and $f_3 = \tau$, equations (2.3.3) and (2.3.5) imply the Corollary. □

Exercise 2.3.5.

(1) Find all curves in $\mathbb{R}^3$ with constant curvature and torsion.
(2) Find all curves in $\mathbb{R}^3$ with constant principal curvatures.
(3) Let α be a curve in $\mathbb{R}^n$ parametrized by arc length. Prove that for generic set of curves α, there exists an o.n. moving frame $(e_1, \cdots, e_n)$ such that

$$(e_1, \cdots, e_n)' = (e_1, \cdots, e_n)P,$$

where

$$P = \begin{pmatrix} 0 & -k_1 & 0 & \cdots & & \\ k_1 & 0 & -k_2 & & & \\ 0 & k_2 & 0 & -k_3 & & \\ & & & & & \\ & & & & 0 & -k_{n-1} \\ 0 & & & & k_{n-1} & 0 \end{pmatrix}$$

for some smooth functions $k_1, \cdots, k_{n-1}$. (So the coefficient matrix for $(e_1, \cdots, e_n)'$ is tri-diangle). This is an analogue of the Frenet frame for curves in $\mathbb{R}^n$.
(4) Formulate and prove the fundamental theorem for curves in $\mathbb{R}^n$ with the frame given in the above exercise.
(5) Let α be a curve in $\mathbb{R}^n$ parametrized by arc length.
   (i) Prove that there exists a o.n. moving frame $(e_1, v_1, \cdots, v_{n-1})$ such that

$$(e_1, v_1, \cdots, v_{n-1})' = (e_1, v_1, \cdots, v_{n-1})P,$$

   where

$$P = \begin{pmatrix} 0 & -k_1 & -k_2 & \cdots & -k_{n-1} \\ k_1 & 0 & \cdots & & \\ k_2 & 0 & \cdots & & \\ \cdot & & & & \\ k_{n-1} & 0 & & & \end{pmatrix}$$

   for some smooth function $k_1, \cdots, k_{n-1}$. This is the parallel frame for curves in $\mathbb{R}^n$.
   (ii) Formulate and prove the fundamental theorem for curves in $\mathbb{R}^n$ with the parallel frame.
   (iii) Find all curves in $\mathbb{R}^n$ with constant principal curvatures.

2.4. **Frobenius Theorem.**

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^2$, and $A, B; \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ smooth maps. Consider the following first order PDE system for $u : \mathcal{O} \to \mathbb{R}$:

$$\begin{cases} \frac{\partial u}{\partial x} = A(x, y, u(x, y)), \\ \frac{\partial u}{\partial y} = B(x, y, u(x, y)). \end{cases} \tag{2.4.1}$$

The Frobenius Theorem gives a necessary and sufficient condition for the first order PDE system (2.4.1) to be solvable. We need to use this theorem extensively in the study of curve evolutions and of surfaces in $\mathbb{R}^3$. We will see from the proof of this theorem that although we are dealing with PDEs, the algorithm to construct solutions of this PDE is to solve one ODE system in x variable and the solve a family of ODE systems in y variables. The condition given in the Frobenius Theorem guarantees that this process produces a solution of the first order PDE.

If the system (2.4.1) admits a smooth solution $u(x, y)$, then the mixed derivatives $u_{xy} = u_{yx}$, i.e.,

$$\begin{aligned} u_{xy} &= (u_x)_y = (A(x, y, u(x, y))_y = A_y + A_u u_y = A_y + A_u B \\ &= (u_y)_x = (B(x, y, u(x, y))_x = B_x + B_u u_x = B_x + B_u A. \end{aligned}$$

So A, B must satisfy the compatibility condition:

$$A_y + A_u B = B_x + B_u A.$$

Moreover, this is also a sufficient condition for system (2.4.1) to be solvable:

Theorem 2.4.1. (Frobenius Theorem) *Let $U_1 \subset \mathbb{R}^2$ and $U_2 \subset \mathbb{R}^n$ be open subsets, $A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n) : U_1 \times U_2 \to \mathbb{R}^n$ smooth maps, $(x_0, y_0) \in U_1$, and $p_0 \in U_2$. Then the following first order system*

$$\begin{cases} \frac{\partial u}{\partial x} = A(x, y, u(x, y)), \\ \frac{\partial u}{\partial y} = B(x, y, u(x, y)), \\ u(x_0, y_0) = p_0, \end{cases} \tag{2.4.2}$$

has a smooth solution for u in a neighborhood of (x_0, y_0) if and only if

$$(A_i)_y + \sum_{j=1}^n \frac{\partial A_i}{\partial u_j} B_j = (B_i)_x + \sum_{j=1}^n \frac{\partial B_i}{\partial u_j} A_j, \qquad 1 \le i \le n. \tag{2.4.3}$$

Proof. If u is a smooth solution of (2.4.2), then $(u_x)_y = (u_y)_x$ implies that the compatibility condition (2.4.3) must hold.

Conversely, assume A, B satisfy (2.4.3), we want to solve (2.4.2). We proceed as follows: The existence and uniqueness of solutions of ODE, Theorem 2.1.2, implies that there exist $\delta > 0$ and $\alpha : (x_0 - \delta, x_0 + \delta) \to U_2$ satisfying

$$\begin{cases} \frac{d\alpha}{dx} = A(x, y_0, \alpha(x)), \\ \alpha(x_0) = p_0. \end{cases} \tag{2.4.4}$$

For each $x \in (x_0 - \delta, x_0 + \delta)$, let $\beta^x(y)$ denote the unique solution of

$$\begin{cases} \frac{d\beta^x}{dy} = B(x, y, \beta^x(y)), \\ \beta^x(y_0) = \alpha(x). \end{cases} \tag{2.4.5}$$

Set $u(x, y) = \beta^x(y)$. By construction, u satisfies the second equation of (2.4.2) and $u(x_0, y_0) = p_0$. It remains to prove u satisfies the first equation of (2.4.2). To prove this, we compute the y-derivative of $z(x, y) = u_x - A(x, y, u(x, y))$ to get

$$\begin{aligned} z_y &= (u_x - A(x, y, u))_y = u_{xy} - A_y - A_u u_y = (u_y)_x - (A_y + A_u B) \\ &= (B(x, y, u))_x - (A_y + A_u B) = B_x + B_u u_x - (A_y + A_u B) \\ &= B_x + B_u u_x - (B_x + B_u A) = B_u(u_x - A) = B_u(x, y, u(x, y))z. \end{aligned}$$

This proves z is a solution of the following differential equation:

$$z_y(x, y) = B_u(x, y, z)z(x, y). \tag{2.4.6}$$

But

$$z(x, y_0) = u_x(x, y_0) - A(x, y_0, u(x, y_0)) = \alpha'(x) - A(x, y_0, \alpha(x)) = 0.$$

Since the constant function zero is also a solution of (2.4.6) with 0 initial data, by the uniqueness of solutions of ODE (Theorem 2.1.2) we have $z = 0$, i.e., u satisfies the second equation of (2.4.2). □

Remark The proof of Theorem 2.4.1 gives an algorithm to construct numerical solution of (2.4.2). The algorithm is as follows: First solve the ODE (2.4.4) on the horizontal line $y = y_0$ by a numerical method (for example Runge-Kutta) to get $u(x_k, y_0)$ for $x_k = x_0 + k\epsilon$ where ϵ is the step size in the numerical method. Then we solve the ODE system (2.4.5) on the vertical line $x = x_n$ for each n to get the value $u(x_n, y_m)$.

Corollary 2.4.2. *Let U_0 be an open subset of $\mathbb{R}^2$, $(x_0, y_0) \in U_0$, $C \in gl(n)$, and $P, Q : U_0 \to gl(n)$ smooth maps. Then the following initial value problem*

$$\begin{cases} g_x = gP, \\ g_y = gQ, \\ g(x_0, y_0) = C \end{cases} \tag{2.4.7}$$

has a $gl(n)$-valued smooth solution g defined in a neighborhood of (x_0, y_0) if and only if

$$P_y - Q_x = [P, Q] = PQ - QP. \tag{2.4.8}$$

Proof. This Corollary follows from Theorem 2.4.1 with $A(x, y, u) = uP(x, y)$ and $B(x, y, u) = uQ(x, y)$. We can also compute the mixed derivatives directly as follows:

$$(g_x)_y = (gP)_y = g_yP + gP_y = gQP + gP_y = g(QP + P_y),$$
$$= (g_y)_x = (gQ)_x = g_xQ + gQ_x = gPQ + gQ_x = g(PQ + Q_x).$$

So the compatibility condition is

$$QP + P_y = PQ + Q_x,$$

which is (2.4.8). □

2.5. **Smoke-ring equation and the non-linear Schrödinger equation.**

In 1906, a graduate student of Levi-Civita, da Rios, wrote a master degree thesis, in which he modeled the movement of a thin vortex by the motion of a curve propagating in $\mathbb{R}^3$ according to:

$$\gamma_t = \gamma_x \times \gamma_{xx}. \tag{2.5.1}$$

It is called the *vortex filament equation* or *smoke ring equation*. This equation is viewed as a dynamical system on the space of curves in $\mathbb{R}^3$.

Proposition 2.5.1. *If $\gamma(x, t)$ is a solution of the smoke ring equation (2.5.1) and $||\gamma_x(x, 0)|| = 1$ for all x, then $||\gamma_x(x, t)|| = 1$ for all (x, t). In other words, if $\gamma(\cdot, 0)$ is parametrized by arc length, then so is $\gamma(\cdot, t)$ for all t.*

So for a solution γ of (2.5.1), we may assume that $\gamma(\cdot, t)$ is parametrized by arc length for all t. Next we explain the geometric meaning of the evolution equation (2.5.1) on the space of curves in $\mathbb{R}^3$. Let $(e_1, e_2, e_3)(\cdot, t)$ denote the Frenet frame of the curve $\gamma(\cdot, t)$. Since $\gamma_x = e_1$ and $\gamma_{xx} = (e_1)_x = ke_2$, the curve flow (2.5.1) becomes

$$\gamma_t = ke_1 \times e_2 = ke_3.$$

In other words, the curve flow (2.5.1) moves in the direction of binormal with curvature as its speed. If we use a parallel frame (e_1, v_1, v_2), then by Corollary 2.3.4

$$ke_3 = k(\sin\beta v_1 + \cos\beta v_2) = -k_2v_1 + k_1v_2.$$

So the curve flow (2.5.1) becomes

$$\gamma_t = -k_2 v_1 + k_1 v_2. \tag{2.5.2}$$

By Proposition 2.3.3, $\alpha'' = k_1 v_1 + k_2 v_2$. So the above equation can be viewed as

$$\gamma_t = J_t(\alpha'') = J_t(k_1 v_1 + k_2 v_2), \tag{2.5.3}$$

where J_t is the rotation of $\frac{\pi}{2}$ on the normal plane at $\alpha(t)$. Note that parallel frame (v_1, v_2) is not unique. Since $\alpha'' = k_1 v_1 + k_2 v_2$ and the rotation of $\frac{\pi}{2}$ in the normal plane is independent of the choice of the parallel frame, the right hand side of equation (2.5.3) is independent of the choice of parallel frame.

Next we want to derive the evolution equation of principal curvatures for $\gamma(\cdot, t)$ for a solution of the smoke-ring equation. We will show that they evolve according to the non-linear Schrödinger equation (NLS), which is the equation models the motion of wave envelope travelling in an optic fiber:

$$q_t = i\left(q_{xx} + \frac{1}{2}|q|^2 q\right). \qquad \text{NLS} \tag{2.5.4}$$

The NLS is one of the model soliton equations. In particular, the Cauchy problem with either rapidly decaying initial data or periodic data have long time existence, and there are infinitely many explicit soliton solutions.

In 1970's, Hasimoto showed that if γ is a solution of the smoke-ring equation (2.5.1), then $q = k\exp(\int \tau dx)$ is a solution of the NLS. We explain the equivalence between the smok-ring equation and the NLS below.

Suppose γ is a solution of (2.5.1). Choose a parallel frame

$$(e_1, v_1, v_2)(\cdot, t)$$

for each curve $\gamma(\cdot, t)$. Let $k_1(\cdot, t)$ and $k_2(\cdot, t)$ denote the principal curvatures of $\gamma(\cdot, t)$ along $v_1(\cdot, t)$ and $v_2(\cdot, t)$ respectively. So we have

$$(e_1, v_1, v_2)_x = (e_1, v_1, v_2)\begin{pmatrix} 0 & -k_1 & -k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{pmatrix} \tag{2.5.5}$$

We want to compute the evolution of (e_1, v_1, v_2), i.e., $(e_1, v_1, v_2)_t$. We first compute

$$\begin{aligned}(e_1)_t = (\gamma_x)_t = (\gamma_t)_x &= (-k_2 v_1 + k_1 v_2)_x \\ &= -(k_2)_x v_1 - k_2 (v_1)_x + (k_1)_x v_2 + k_1 (v_2)_x, \quad \text{by}(2.5.5) \\ &= -(k_2)_x v_1 - k_2(-k_1 e_1) + (k_1)_x v_2 + k_1(-k_2 e_1) \\ &= -(k_2)_x v_1 + (k_1)_x v_2.\end{aligned}$$

Since (e_1, v_1, v_2) is o.n., by Proposition 2.1.1 there exists a function u so that

$$(e_1, v_1, v_2)_t = (e_1, v_1, v_2) \begin{pmatrix} 0 & (k_2)_x & -(k_1)_x \\ -(k_2)_x & 0 & -u \\ (k_1)_x & u & 0 \end{pmatrix},$$

or equivalently,

$$\begin{cases} (e_1)_t = -(k_2)_x v_1 + (k_1)_x v_2, \\ (v_1)_t = (k_2)_x e_1 + u v_2, \\ (v_2)_t = -(k_1)_x e_1 - u v_1. \end{cases}$$

We claim that

$$u_x = -\frac{1}{2}(k_1^2 + k_2^2)_x.$$

To prove this, we compute

$$\begin{aligned}(v_1)_{xt} \cdot v_2 &= (-k_1 e_1)_t \cdot v_2 = -k_1 (e_1)_t \cdot v_2 \\ &= -k_1(-(k_2)_x v_1 + (k_1)_x v_2) \cdot v_2 = -k_1 (k_1)_x, \\ = (v_1)_{tx} \cdot v_2 &= ((k_2)_x e_1 + u v_2)_x \cdot v_2 = (k_2)_x (e_1)_x \cdot v_2 + u_x \\ &= (k_2)_x (k_1 v_1 + k_2 v_2) \cdot v_2 + u_x = (k_2)_x k_2 + u_x.\end{aligned}$$

This proves the claim. Hence

$$u(x,t) = -\frac{1}{2}(k_1^2 + k_2^2) + c(t)$$

for some smooth function $c(t)$. Remember that for each fixed t, we can rotate $(v_1, v_2)(\cdot, t)$ by a constant angle $\theta(t)$ to another parallel normal frame $(\tilde{v}_1, \tilde{v}_2)$ of $\gamma(\cdot, t)$, i.e.,

$$\begin{cases} \tilde{v}_1(x,t) = \cos\theta(t) v_1(x,t) + \sin\theta(t) v_2(x,t), \\ \tilde{v}_2(x,t) = -\sin\theta(t) v_1(x,t) + \cos\theta(t) v_2(x,t). \end{cases}$$

A direct computation shows that

$$\tilde{u} = (\tilde{v}_1)_t \cdot \tilde{v}_2 = u + \frac{d\theta}{dt} = -\frac{1}{2}(k_1^2 + k_2^2) + c(t) + \theta'(t).$$

If we choose θ so that $\theta' = -c$, then the new parallel frame satisfies

$$(e_1, \tilde{v}_1, \tilde{v}_2)_t = (e_1, \tilde{v}_1, \tilde{v}_2) \begin{pmatrix} 0 & (\tilde{k}_2)_x & -(\tilde{k}_1)_x \\ -(\tilde{k}_2)_x & 0 & \frac{\tilde{k}_1^2+\tilde{k}_2^2}{2} \\ (\tilde{k}_1)_x & -\frac{\tilde{k}_1^2+\tilde{k}_2^2}{2} & 0 \end{pmatrix}.$$

So we have proved the first part of the following Theorem:

Theorem 2.5.2. *Suppose $\gamma(x,t)$ is a solution of the smoke ring equation (2.5.1) and $||\gamma_x(x,0)|| = 1$ for all x. Then*

(1) *there exists a parallel normal frame $(v_1, v_2)(\cdot, t)$ for each curve $\gamma(\cdot, t)$ so that*

$$\begin{cases} g_x = g \begin{pmatrix} 0 & -k_1 & -k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{pmatrix} \\ g_t = g \begin{pmatrix} 0 & (k_2)_x & -(k_1)_x \\ -(k_2)_x & 0 & \frac{k_1^2+k_2^2}{2} \\ (k_1)_x & -\frac{k_1^2+k_2^2}{2} & 0 \end{pmatrix}, \end{cases} \tag{2.5.6}$$

where $g = (e_1, v_1, v_2)$ and $k_1(\cdot, t)$ and $k_2(\cdot, t)$ are the principal curvatures of $\gamma(\cdot, t)$ with respect to $v_1(\cdot, t), v_2(\cdot, t)$ respectively,

(2) *$q = k_1 + ik_2$ is a soluton of the NLS (2.5.4).*

Proof. We have proved (1). For (2), we use (2.5.6) to compute the evolution of $(k_1)_t$ and $(k_2)_t$:

$$\begin{aligned} (k_1)_t &= ((e_1)_x \cdot v_1)_t = (e_1)_{xt} \cdot v_1 + (e_1)_x \cdot (v_1)_t \\ &= (e_1)_{tx} \cdot v_1 + (k_1 v_1 + k_2 v_2) \cdot \left((k_2)_x e_1 - \left(\frac{k_1^2 + k_2^2}{2} \right) v_2 \right) \\ &= (-(k_2)_x v_1 + (k_1)_x v_2)_x \cdot v_1 - \frac{k_2(k_1^2 + k_2^2)}{2} \\ &= -(k_2)_{xx} - \frac{k_2(k_1^2 + k_2^2)}{2}. \end{aligned}$$

Similar computation gives

$$(k_2)_t = (k_1)_{xx} + \frac{k_1(k_1^2 + k_2^2)}{2}.$$

Let $q = k_1 + ik_2$. Then we have

$$q_t = i(q_{xx} + \frac{|q|^2}{2}\, q).$$

□

The converse of Theorem 2.5.2 is also true. Given a solution $q = k_1 + ik_2$ of the NLS (2.5.4), we need to first solve the first order PDE system (2.5.6) to get (e_1, v_1, v_2), then solve another first order PDE system

$$\begin{cases} \gamma_x = e_1, \\ \gamma_t = -k_2 v_1 + k_1 v_2. \end{cases} \tag{2.5.7}$$

Then $\gamma(x,t)$ satisfies the smok-ring equation (2.5.1). Now we are ready to prove the converse of Theorem 2.5.2. Let $q = k_1 + ik_2$ be a solution of NLS. To construct a solution $\gamma(x,t)$ of the smoke ring equation (2.5.1), we need to solve two first order PDE systems (2.5.6) and (2.5.7). System (2.5.6) is of the form (2.4.7). By Corollary 2.4.2, the compatibility condition is (2.4.8), i.e.,

$$P_t - Q_x = [P, Q], \tag{2.5.8}$$

where

$$P = \begin{pmatrix} 0 & -k_1 & -k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & (k_2)_x & -(k_1)_x \\ -(k_2)_x & 0 & \frac{k_1^2+k_2^2}{2} \\ (k_1)_x & -\frac{k_1^2+k_2^2}{2} & 0 \end{pmatrix}$$

Since P, Q are skew-symmetric,

$$[P, Q]^t = (PQ - QP)^t = (Q^t P^t - P^t Q^t) = (QP - PQ) = -[P, Q].$$

Hence $[P, Q]$ is is skew-symmetric. So we only need to check the 21, 31 and 32 entries of (2.5.8). A direct computation shows that (2.5.8) holds if and only if

$$\begin{cases} (k_1)_t + (k_2)_{xx} = -\frac{k_2(k_1^2+k_2^2)}{2}, \\ (k_2)_t - (k_1)_{xx} = \frac{k_1(k_1^2+k_2^2)}{2}, \end{cases}$$

which is the NLS equation for $q = k_1 + ik_2$. So by Corollary 2.4.2, given a constant orthogonal matrix (u_1^0, u_2^0, u_3^0), there exists a unique solution $g = (e_1, v_1, v_2)$ for system (2.5.6) with initial data $g(0) = (u_1^0, u_2^0, u_3^0)$. To construct the solution γ for the smoke ring equation, we need to solve (2.5.7). The compatibility condition for (2.5.7) is given by $\gamma_{xt} = \gamma_{tx}$, which is

$$(e_1)_t = (-k_2 v_1 + k_1 v_2)_x.$$

This holds because (e_1, v_1, v_2) satisfies (2.5.6). We carry out this direct computation:

$$\begin{aligned}(e_1)_t &= -(k_2)_x v_1 + (k_1)_x v_2,\\ (-k_2 v_1 + k_1 v_2)_x &= -(k_2)_x v_1 - k_2 (v_1)_x + (k_1)_x v_2 + k_1 (v_2)_x\\ &= -(k_2)_x v_1 + k_2 k_1 e_1 + (k_1)_x v_2 - k_1 (k_2 e_1)\\ &= -(k_2)_x v_1 + (k_1)_x v_2.\end{aligned}$$

In other word, we have proved

Theorem 2.5.3. *Let $q = k_1 + ik_2$ be a solution of NLS, $p_0 \in \mathbb{R}^3$, and (u_1, u_2, u_3) a constant orthogonal matrix. Then there exist a solution γ of the smoke ring equation and a parallel frame $(e_1, v_1, v_2)(\cdot, t)$ for $\gamma(\cdot, t)$ so that $k_1(\cdot, t), k_2(\cdot, t)$ are the principal curvatures along $v_1(\cdot, t), v_2(\cdot, t)$ respectively, $\gamma(0, 0) = p_0$, and $(e_1, v_1, v_2)(0, 0) = (u_1, u_2, u_3)$.*

Note that given a solution $q = k_1 + ik_2$ of NLS, the proof of Theroem 2.5.3 gives an algorithm to construct numerical solution of the smoke-ring equation.

3. Surfaces in $\mathbb{R}^3$

3.1. Some linear algebra.

Let V be a vector space, and $(\ ,)$ an inner product on V. A linear map $T: V \to V$ is *self-adjoint* if

$$(A(v_1), v_2) = (v_1, A(v_2)) \quad \forall \ \ v_1, v_2 \in V.$$

The following is the Spectral Theorem for self-adjoint operators:

Theorem 3.1.1. *Suppose V is an n-dimensional vector space equipped with an inner product $(,)$. If $T: V \to V$ is a linear self-adjoint operator, then*

1. *eigenvalues of T are real,*
2. *there is an orthonormal basis $v_1, \cdots, v_n$ of V that are eigenvectors of T, i.e., $T(v_i) = \lambda_i v_i$ for $1 \leq i \leq n$.*

A bilinear functional $b: V \times V \to \mathbb{R}$ is *symmetric* if

$$b(v, w) = b(v, w) \qquad \text{for all} \ \ v, w \in V.$$

A symmetric bilinear form b is *positive definite* if $b(v, v) > 0$ for all non-zero $v \in V$.

We associate to each self-adjoint linear map T on V a symmetric bilinear functional:

Theorem 3.1.2. *Let $T: V \to V$ be a self-adjoint linear map, and $b: V \times V \to \mathbb{R}$ the map defined by*

$$b(v_1, v_2) = (T(v_1), v_2).$$

Then

1. *b is a symmetric bilinear form on V,*
2. *all eigenvalues of T are positive if and only if b is positive definite.*

Given a linear map $T: V \to V$ and a basis $v_1, \cdots, v_n$ of T, we can associate to T a matrix A so that T is represented by the multiplication of A. This is because each $T(v_i)$ can be written as a linear combination of the basis:

$$T(v_i) = \sum_j a_{ji} v_j.$$

Write $v = \sum_j x_j v_j$, and $T(v) = \sum_j y_j v_j$. A direct computation gives $y = Ax$, where $A = (a_{ij})$, $x = (x_1, \cdots, x_n)^t$, and $y = (y_1, \cdots, y_n)^t$. In other words, after choosing a basis of V, the operator T looks like the map $x \mapsto Ax$. We call A *the matrix of T associated to the basis*

$v_1, \cdots, v_n$. If we change basis, the corresponding matrix of T change by a conjugation:

Proposition 3.1.3. *Let $T : V \to V$ be a linear map, and A and B the matrices of T associated to bases $\{v_1, \cdots, v_n\}$ and $\{u_1, \cdots, u_n\}$ of V respectively. Write $u_i = \sum_j g_{ji} v_j$. Then $B = g^{-1}Ag$, where $g = (g_{ij})$.*

Below we list some properties of determinants:

$$\det(AB) = \det(A)\det(B),$$
$$\det(A^t) = \det(A),$$
$$\det(gAg^{-1}) = \det(A).$$

Let A denote the matrix associated to the linear operator $T : V \to V$ with respect to the basis $v_1, \cdots, v_n$. Define

$$\mathrm{tr}(T) = \mathrm{tr}(A), \quad \det(T) = \det(A),$$

where A is the matrix of T associated to the basis $v_1, \cdots, v_n$. Since

$$\mathrm{tr}(gAg^{-1}) = \mathrm{tr}(A), \quad \det(gAg^{-1}) = \det(A),$$

$\mathrm{tr}(T)$ and $\det(T)$ are independent of the choice of basis. Moreover, if T has an eignbasis with eigenvalues $\lambda_1, \cdots, \lambda_n$, then

$$\mathrm{tr}(T) = \sum_{j=1}^{n} \lambda_j, \qquad \det(T) = \prod_{j=1}^{n} \lambda_j.$$

Let $v_1, \cdots, v_n$ be a basis of V, $b : V \times V \to \mathbb{R}$ a symmetric bilinear form, and $b_{ij} = b(v_i, v_j)$. Given $\xi = \sum_i x_i v_i$ and $\eta = \sum_i y_i v_i$, we have

$$b(\xi, \eta) = \sum_{ij} b_{ij} x_i y_j = X^t BY,$$

where $X^t = (x_1, \ldots, x_n)$, $Y^t = (y_1, \ldots, y_n)$, and $B = (b_{ij})$. We call $B = (b_{ij})$ the matrix associated to b with respect to basis $v_1, \cdots, v_n$.

Proposition 3.1.4. *Let $(V, (\ ,\ ))$ be an inner product space, $T : V \to V$ a self-adjoint operator, $b(X, Y) = (T(X), Y)$ the symmetric bilinear form associated to T, and $A = (a_{ij})$, $B = (b_{ij})$ the matrices of T and b associated to basis $v_1, \cdots, v_n$ respectively. Let $g_{ij} = (v_i, v_j)$, $G = (g_{ij})$, and $G^{-1} = (g^{ij})$ the inverse of G. Then*

(1) $B = A^t G$,
(2) $A = G^{-1}B$,
(3) $\det(A) = \frac{\det(B)}{\det(G)}$, $\mathrm{tr}(A) = \sum_{i,j} b_{ij} g^{ij}$.

Proof. Compute directly to get

$$b_{ij} = (T(v_i), v_j) = (\sum_k a_{ki} v_k, v_j) = \sum_k a_{ki} g_{kj} = (A^t G)_{ij}.$$

This proves (1). Statement (2) and (3) are consequences of (1). □

Suppose $b : V \times V \to \mathbb{R}$ is the symmetric bilinear form associated to the self-adjoint operator $T : V \to V$, and $v_1, \cdots, v_n$ are orthonormal eigenbasis of T with eigenvalues $\lambda_1, \cdots, \lambda_n$. Then the diagonal matrix $\mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ is the matrix associated to both T and b with respect to the basis $v_1, \cdots, v_n$. Moreover, given $v \in V$, write $v = \sum_{i=1}^n x_i v_i$, then

$$b(v, v) = b\left(\sum_{i=1}^n x_i v_i, \sum_{j=1}^n x_j v_j\right) = \left(\sum_{i=1}^n \lambda_i x_i v_i, \sum_{j=1}^n x_j v_j\right) = \sum_{i=1}^n \lambda_i x_i^2.$$

In other words, the quadratic form $Q(v) = b(v, v)$ associated to b is diagonalized. We also see that the maximum (minimum resp.) of $b(v, v)$ taken among all $\|v\| = 1$ is the largest (smallest resp.) eigenvalue of T.

Tensor algebra

Let V be a finite dimensional vector space, and the dual $V^* = L(V, \mathbb{R})$ the space of all linear functional from V to $\mathbb{R}$. Note that V is naturally isomorphic to $V^{**} = (V^*)^* = L(V^*, \mathbb{R})$ via the map

$$\phi(v)(\ell) = \ell(v), \quad v \in V, \quad \ell \in V^*.$$

The *tensor product* $V_1^* \otimes \cdots V_k^*$ is the vector space of all multilinear functionals $b : V_1 \times \cdots \times V_k \to \mathbb{R}$, and the tensor product

$$\mathcal{T}_m^k(V) = \otimes^k V^* \otimes^m V$$

is the space of all multilinear functionals from $V \times \cdots V \times V^* \times \cdots \times V^*$ to $\mathbb{R}$ (here we use k copies of V and m copies of V^*).

Given $f_i \in V_i^*$ and $\xi_{i+k} \in V_{i+k}$, let $f_1 \otimes \cdots \otimes f_k \otimes \xi_{k+1} \otimes \cdots \otimes \xi_{k+m}$ be the element in $V_1^* \otimes V_k^* \otimes V_{k+1} \otimes \cdots \otimes V_{k+m}$ defined by

$$\begin{aligned}(f_1 \otimes \cdots \otimes f_k \otimes \xi_{k+1} \otimes \cdots \otimes \xi_{k+m})(u_1, \cdots, u_k, h_{k+1}, \cdots, h_{k+m}) \\ = f_1(u_1) \cdots f_k(u_k) h_{k+1}(\xi_{k+1}) \cdots h_{k+m}(\xi_{k+m})\end{aligned}$$

for $u_i \in V_i$ and $h_{k+i} \in V_{k+i}^*$.

Proposition 3.1.5. *Let $v_1, \cdots, v_n$ be a basis of V, and $\ell_1, \cdots, \ell_n \in V^*$ the dual basis, i.e., $\ell_i(v_j) = \delta_{ij}$. Let $b : V \times V \to \mathbb{R}$ be a bilinear functional, and $b_{ij} = b(v_i, v_j)$. Then*

$$b = \sum_{ij} b_{ij}\ \ell_i \otimes \ell_j.$$

Let $L(V_1, V_2)$ denote the space of all linear maps from V_1 to V_2. Given $\ell \in V_1^*$ and $v \in V_2$, the tensor product $\ell \otimes v$ can be identified as the linear map from V_1 to V_2 defined by $u \mapsto \ell(u)v$. It is easy to check that

$$L(V_1, V_2) \simeq V_1^* \otimes V_2.$$

In fact,

Proposition 3.1.6. *Let $\{v_1, \cdots, v_n\}$ and $\{w_1, \cdots, w_m\}$ be basis of V_1 and V_2 respectively, and $\ell_1, \cdots, \ell_n$ the dual basis of V_1^*. Suppose $T : V_1 \to V_2$ is a linear map, and $T(v_i) = \sum_{j=1}^m a_{ji} w_j$ for $1 \le i \le n$. Then*

$$T = \sum_{ij} a_{ji}\ \ell_i \otimes w_j.$$

Corollary 3.1.7. *Suppose $\{v_1, \cdots, v_n\}$ is a basis of V and $\{\ell_1, \cdots, \ell_n\}$ is the dual basis of V^*. Then the identity map $\mathrm{Id} = \sum_{i=1}^n \ell_i \otimes e_i$.*

A multilinear functional $u \in \otimes^k(V^*)$ is called *alternating* if

$$u(\xi_{\tau(1)}, \cdots, \xi_{\tau(k)}) = \mathrm{sgn}\ (\tau) u(\xi_1, \cdots, \xi_k)$$

for all $\xi_1, \cdots, \xi_k \in V$, where τ a permutation of $\{1, \cdots, k\}$ and $\mathrm{sgn}(\tau)$ is the sign of τ.

For $\theta_1, \theta_2 \in V^*$, let $\theta_1 \wedge \theta_2$ denote the bilinear functional on V defined by

$$(\theta_1 \wedge \theta_2)(\xi_1, \xi_2) = \theta_1(\xi_1)\theta_2(\xi_2) - \theta_2(\xi_1)\theta_1(\xi_2).$$

By definition, $\theta_1 \wedge \theta_2$ is alternating. So $\theta_1 \wedge \theta_2$ is the anti-symmetrization of $\theta_1 \otimes \theta_2$.

Let $\wedge^k(V^*)$ denote the space of all alternating functionals in $\otimes^k(V^*)$. It follows from definitions that we have

Proposition 3.1.8. *The following are true:*

1. *If $\theta \in V^*$, then $\theta \wedge \theta = 0$.*
2. *If $\theta_1, \theta_2 \in V^*$, then $\theta_1 \wedge \theta_2 = -\theta_2 \wedge \theta_1$.*
3. *If $\ell_1, \cdots, \ell_n$ is a basis of V^*, then*

$$\{\ell_{i_1} \wedge \cdots \wedge \ell_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\}$$

is a basis of $\wedge^k(V^*)$. In particular, $\dim(\wedge^n(V^*)) = 1$ and $\dim(\wedge^k(V^*)) = 0$ if $k > n$.

3.2. **Embedded surfaces in $\mathbb{R}^3$.**

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^2$. A smooth map $f : \mathcal{O} \to \mathbb{R}^3$ is called an *immersion* if $f_{x_1}(p), f_{x_2}(p)$ are linearly independent for all $p \in \mathcal{O}$.

Let M be a subset of $\mathbb{R}^3$. A subset U of M is *open in the induced topology* if there exists an open subset U_0 of $\mathbb{R}^n$ so that $U = U_0 \cap M$.

Definition 3.2.1. A subset M of $\mathbb{R}^3$, equipped with the induced topology, is called an *embedded surface* if there exist an open cover $\{U_\alpha \mid \alpha \in I\}$ of M and a collection of homeomorphisms $\phi_\alpha : \mathcal{O}_\alpha \to U_\alpha$ from open subset $\mathcal{O}_\alpha$ of $\mathbb{R}^2$ onto U_α such that

(1) $\phi_\alpha : \mathcal{O}_\alpha \to U_\alpha \subset \mathbb{R}^3$ is an immersion,
(2) $\phi_\beta^{-1}\phi_\alpha : \phi_\alpha^{-1}(U_\alpha \cap U_\beta) \to \phi_\beta^{-1}(U_\alpha \cap U_\beta)$ is a smooth diffomorphism for any $\alpha, \beta \in I$.

The collection $\mathcal{A} = \{\phi_\alpha : \mathcal{O}_\alpha \to U_\alpha \mid \alpha \in I\}$ is called an *atlas* of M. A homeomorphism $\phi : \mathcal{O} \to U \subset M$ is said to be *compatible to* $\mathcal{A}$ if

$$\phi_\alpha^{-1} \circ \phi : \phi^{-1}(U \cap U_\alpha) \to \phi_\alpha^{-1}(U \cap U\alpha)$$

is a diffeomorphism for all $\alpha \in I$. Such ϕ is called a *local coordinate system* or a *local chart* on M.

Example 3.2.2. Examples of embedded surfaces in $\mathbb{R}^3$

(1) *Graph.*

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^2$, and $u : \mathcal{O} \to \mathbb{R}$ a smooth proper map. Here a map is *proper* if the preimage of any compact subset is compact. The graph

$$M = \{(x_1, x_2, u(x_1, x_2)) \mid x_1, x_2 \in \mathcal{O}\}$$

is an embedded surface in $\mathbb{R}^3$ with one chart defined by $\phi(x_1, x_2) = (x_1, x_2, u(x_1, x_2))$.

(2) *Surface of revolution.*

Let $\alpha(s) = (y(s), z(s)) : (a, b) \to \mathbb{R}^2$ be the parametrization of a smooth curve C in the yz-plane, and M the set obtained by rotating the curve C along the y-axis, i.e.,

$$M = \{(z(s)\cos t, y(s), z(s)\sin t) \mid s \in (a, b), t \in [0, 2\pi]\}.$$

It is easy to check that M is an embedded surface in $\mathbb{R}^3$ (we need two charts to cover M).

(3) Let $\mathcal{O}$ be an open subset of $\mathbb{R}^3$, and $f : \mathcal{O} \to \mathbb{R}^3$ be a smooth map such that f_{x_1}, f_{x_2} are linearly independent at every point of $\mathcal{O}$ and f is a homeomorphism from $\mathcal{O}$ to $f(\mathcal{O})$, where $f(\mathcal{O})$

is equipped with the induced topology. Then $M = f(\mathcal{O})$ is an embedded surface of $\mathbb{R}^3$.

(4) If M is an embedded surface of $\mathbb{R}^3$ and $\phi_\alpha : \mathcal{O}_\alpha \to U_\alpha$ is a local coordinate system on M, then U_α is also an embedded surface in $\mathbb{R}^3$.

The following Proposition is a consequence of the Implicit Function Theorem:

Proposition 3.2.3. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. If $c \in \mathbb{R}$ such that for all $p \in f^{-1}(c)$ the gradient*

$$\nabla f(p) = (f_{x_1}(p), f_{x_2}(p), f_{x_3}(p)) \neq 0,$$

then $M = f^{-1}(c)$ is an embedded surface.

Applying the above Proposition to the function

$$f(x) = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2,$$

we see that the spheres, ellipsoids, hyperbolids are embedded surfaces. Apply to

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2$$

to see that the cylinder is an embedded surface.

Definition 3.2.4.

(1) Let M be an embedded surface in $\mathbb{R}^3$. A map $\alpha : (a, b) \to M$ is called a smooth curve if $\pi_i \circ \alpha : (a, b) \to \mathbb{R}$ is smooth for all $1 \leq i \leq 3$, where $\pi_i : \mathbb{R}^3 \to \mathbb{R}$ is the projection of $\mathbb{R}^3$ to the i-th component.

(2) Let M and N be embedded surfaces in $\mathbb{R}^3$. A map $f : M \to N$ is *smooth* if whenever α is a smooth curve on M implies that $f \circ \alpha$ is a smooth curve on N.

Tangent plane

Let M be an embedded surface in $\mathbb{R}^3$, and $p_0 \in M$. If $\alpha : (-\delta, \delta) \to M$ is a smooth curve with $\alpha(0) = p_0$, then α can be viewed as a map from $(-\delta, \delta)$ to $\mathbb{R}^3$. So we can compute $\alpha'(0)$ as in calculus. The space

$$TM_{p_0} = \{\alpha'(0) \mid \alpha : (-\delta, \delta) \to M \text{ is smooth with } \alpha(0) = p_0\}$$

is a vector space, and is called the *tangent plane of M at p_0*.

If $f : \mathcal{O} \to U \subset M$ is a local coordinate system of M, then $f_{x_1}(q), f_{x_2}(q)$ spans $TM_{f(q)}$ and $\frac{f_{x_1} \times f_{x_2}}{||f_{x_1} \times f_{x_2}||}(q)$ is a unit vector normal to $TM_{f(q)}$.

The differential of a smooth map

Suppose M and N are two embedded surfaces in $\mathbb{R}^3$, and $f : M \to N$ is a smooth map. The *differential of f at p*, df_p, is the linear map from TM_p to $TN_{f(p)}$ defined by

$$df_p(\alpha'(0)) = (f \circ \alpha)'(0),$$

where $\alpha : (-\delta, \delta) \to M$ is a smooth curve on M with $\alpha(0) = p$. It follows from the Chain rule that df_p is well-defined, i.e., if α, β are two curves on M so that $\alpha(0) = \beta(0) = p$ and $\alpha'(0) = \beta'(0)$, then $(f \circ \alpha)'(0) = (f \circ \beta)'(0)$.

Smooth vector fields

Let U be an open subset of M. A map $\xi : U \to \mathbb{R}^3$ is called a *smooth vector field of M defined on M* if

(1) $\xi(p) \in TM_p$ for each $p \in U$,
(2) given any local coordinate system $\phi_\alpha : \mathcal{O}_\alpha \to U_\alpha$ of M there exist smooth function ξ_1, ξ_2 on $\phi_\alpha^{-1}(U \cap U_\alpha)$ so that

$$\xi(f(q)) = \xi_1(q)(\phi_\alpha)_{x_1}(q) + \xi_2(q)(\phi_\alpha)_{x_2}(q)$$

for all $q \in U \cap U_\alpha$.

We will sometimes write a shorthand for ξ in local coordinate system ϕ_α by

$$\xi = \xi_1(\phi_\alpha)_{x_1} + \xi_2(\phi_\alpha)_{x_2}.$$

If $f : \mathcal{O} \to U \subset M$ is a local coordinate system, then we often let $\frac{\partial}{\partial x_i}$ to denote the vector field $f_{x_i} \circ f^{-1}$, i.e.,

$$\frac{\partial}{\partial x_i}(p) := f_{x_i}(f^{-1}(p)).$$

Then the above vector field ξ can be written as $\xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2}$.

Smooth differential 1-forms

Let $f : \mathcal{O} \to U \subset M$ be a local coordinate system on the embedded surface M of $\mathbb{R}^3$. Let $(dx_1)_p, (dx_2)_p$ denote the base of TM_p^* dual to the base $f_{x_1}(p), f_{x_2}(p)$ of TM_p, i.e., $dx_i(f_{x_j}) = \delta_{ij}$. Let $(dx_1 \wedge dx_2)_p = (dx_1)_p \wedge (dx_2)_p$.

A *smooth differential* 1*-form* θ on an open subset U of M is an assignment $\theta(p) \in TM_p^*$ for each $p \in U$ satisfying the following condition:

given any local coordinate system $\phi_\alpha : \mathcal{O}_\alpha \to U_\alpha \subset M$, there exist smooth $A_1, A_2 : \phi_\alpha^{-1}(U \cap U_\alpha) \to \mathbb{R}$ so that

$$\theta_{f(q)} = A_1(q)(dx_1)_{f(q)} + A_2(q)(dx_2)_{f(q)}$$

for all $q \in \phi_\alpha^{-1}(U \cap U_\alpha)$. We often write the above equation in local coordinate system ϕ_α as

$$\theta = A_1 dx_1 + A_2 dx_2.$$

If $h : M \to \mathbb{R}$ is a smooth map, then $dh_p : TM_p \to T\mathbb{R}_{h(p)} = \mathbb{R}$. So we can identify $dh_p \in TM_p^*$ and dh as a smooth differential 1-form.

Differential 2-forms

A *smooth differential* 2*-form* ω on an open subset U of M is an assignment $p \in U \mapsto \omega_p \in \wedge^2 TM_p^*$ satisfying the condition that given any local coordinate system $f : \mathcal{O} \to U_0 \subset M$ there exists a smooth function $h : f^{-1}(U \cap U_0) \to \mathbb{R}$ such that ω can be written as

$$\omega(f(q)) = h(q)(dx_1)_{f(q)} \wedge (dx_2)_{f(q)}$$

for all $q \in f^{-1}(U \cap U_0)$. We will use the notation

$$\omega = h \; dx_1 \wedge dx_2$$

on $U \cap U_0$ with respect to the local coordinate system f.

Wedge products

Let θ_1, θ_2 be two smooth 1-forms on M. The wedge product $\theta_1 \wedge \theta_2$ is defined by

$$\theta_1 \wedge \theta_2(v_1, v_2) = \theta_1(v_1)\theta_2(v_2) - \theta_2(v_1)\theta_1(v_2)$$

for all $v_1, v_2 \in TM_p$ and $p \in M$. It follows from the definition that if θ_1, θ_2 are 1-forms, then

$$\theta_1 \wedge \theta_2 = -\theta_2 \wedge \theta_1.$$

Exterior differentiation

If 1-form θ in local coordinates is $\theta = h_1 dx_1 + h_2 dx_2$, then the *exterior differentiation* of θ is defined to be

$$d\theta = (-(h_1)_{x_2} + (h_2)_{x_1}) \; dx_1 \wedge dx_2.$$

The definition of d is independent of the choice of local coordiante systems. In other words, if $\theta = h_1 dx_1 + h_2 dx_2$ and $\theta = b_1 dy_1 + b_2 dy_2$

with respect to local coordinate systems ϕ_α and ϕ_β respectively, then a direct computation implies that

$$(-(h_1)_{x_2} + (h_2)_{x_1})\ dx_1 \wedge dx_2 = (-(b_1)_{y_2} + (b_2)_{y_1})\ dy_1 \wedge dy_2.$$

A 1-form θ is called *exact* if there exists a smooth function h on M so that $\theta = dh$. A 1-form θ is called *closed* if $d\theta = 0$.

Next we prove the Poincaré Lemma.

Proposition 3.2.5. *Let θ be a 1-form on an embedded surface M in $\mathbb{R}^3$. Then*

(i) *θ is exact implies θ is closed,*
(ii) *if θ is closed, then given any $p \in M$, there exist an open subset U containing p and a smooth function $h : U \to \mathbb{R}$ such that $\theta = dh$ on U, i.e., θ is locally exact.*

Proof. Let $h : M \to \mathbb{R}$ be a smooth function, then by definition of d,

$$d(dh) = d(h_{x_1} dx_1 + h_{x_2} dx_2) = (-(h_{x_1})_{x_2} + (h_{x_2})_{x_1})\ dx_1 \wedge dx_2.$$

So dh is closed. This proves (i).

Let θ be a smooth 1-form. Given a local coordinate system $f : \mathcal{O} \to U_0$ around $p = f(x_1^0, x_2^0)$, we can write $\theta = A(x_1, x_2)dx_1 + B(x_1, x_2)dx_2$ with some smooth function $A, B : \mathcal{O} \to \mathbb{R}$. The closeness of θ means $A_{x_2} = B_{x_1}$. This implies that there exists a small open subset $\mathcal{O}_0$ of $\mathcal{O}$ containing (x_1^0, x_2^0) so that

$$h_{x_1} = A, \quad h_{x_2} = B.$$

So $\theta = dh$ on $U_0 = f(\mathcal{O}_0)$. □

It follows from definitions of d and wedge product that we have

Proposition 3.2.6. *If θ is a 1-form and $h : M \to \mathbb{R}$ smooth, then*

$$d(h\theta) = dh \wedge \theta + h d\theta = d(\theta h) = h d\theta - \theta \wedge dh.$$

Smooth bilinear forms

A *smooth bilinear form b on an open subset U of M* is a collection of bilinear forms $b_p : TM_p \times TM_p \to \mathbb{R}$ with $p \in U$ satisfying the condition that given any local coordinate system $f : \mathcal{O} \to U_0$ on M there exist smooth functions b_{ij} on $f^{-1}(U \cap U_0)$ so that

$$b_{f(q)} = \sum_{i,j=1}^{2} b_{ij}(q)\ (dx_i)_{f(q)} \otimes (dx_j)_{f(q)}.$$

We will write b locally as $\sum_{i,j=1}^2 b_{ij}(x)dx_i \otimes dx_j$. If b_p is symmetric bilinear for each $p \in U$, then $b_{ij} = b_{ji}$. When b is symmetric, we will skip the $\otimes$ and simply write b as $\sum_{i,j=1}^2 b_{ij}(x)dx_i dx_j$.

When we calculate vector fields and differential forms on M, we only need to do the computation in local coordinates of M.

3.3. **The first and second fundamental forms.**

Let M be an embedded surface in $\mathbb{R}^3$, and $f : \mathcal{O} \to U$ a local coordinate system on M. Then f_{x_1}, f_{x_2} form a base of $TM_{f(x)}$ for all $x \in \mathcal{O}$ and

$$N = \frac{f_{x_1} \times f_{x_2}}{\sqrt{\|f_{x_1} \times f_{x_2}\|}}.$$

is the unit vector that is normal to $TM_{f(x_1,x_2)}$.

We will assciate to each $p \in M$ two smooth symmetric bilinear forms, the first and second fundamental forms of M. The *first fundamental form* of M at $p \in M$ denoted by

$$\mathrm{I}_p : TM_p \times TM_p \to \mathbb{R}$$

is the bilinear functional defined by $\mathrm{I}_p(u, v) = u \cdot v$, the dot product of u, v in $\mathbb{R}^3$. Then I_p defines an inner product on TM_p. Let

$$g_{11} = f_{x_1} \cdot f_{x_1}, \quad g_{12} = g_{21} = f_{x_1} \cdot f_{x_2}, \quad g_{21} = f_{x_2} \cdot f_{x_2}.$$

Then

$$\begin{aligned}\mathrm{I} &= g_{11}dx_1 \otimes dx_2 + g_{12}(dx_1 \otimes dx_2 + dx_2 \otimes dx_1) + g_{22}dx_2 \otimes dx_2 \\ &= g_{11}dx_1^2 + 2g_{12}dx_1dx_2 + g_{22}dx_2^2.\end{aligned}$$

Since g_{ij} are smooth functions, I is a smooth symmetric bilinear form on M.

The unit normal vector field N is a map from M to S^2, which is called the *Gauss map* of M. Recall that the differential of N at p is the linear map from TM_p to $T(S^2)_{N(p)}$ defined by

$$dN_p(u) = (N \circ \alpha)'(0),$$

where $\alpha : (-\epsilon, \epsilon) \to M$ is a smooth curve with $\alpha(0) = p$ and $\alpha'(0) = u$. It follows from the definition of dN_p that

$$dN(f_{x_1}) = N_{x_1}, \quad dN(f_{x_2}) = N_{x_2}.$$

Since the tangent plane of S^2 at a point $q \in S^2$ is the plane perpendicular to q. We see that both $TS^2_{N(p)}$ and TM_p are perpendicular to N_p, so they are the same plane. Hence dN_p can be viewed as a linear operator from TM_p to TM_p.

Proposition 3.3.1. *The linear map $dN_p : TM_p \to TM_p$ is self-adjoint.*

Proof. Use $f_{x_i} \cdot N = 0$ to compute

$$dN_p(f_{x_i}) \cdot f_{x_j} = N_{x_i} \cdot f_{x_j} = (N \cdot f_{x_j})_{x_i} - N \cdot f_{x_j x_i} = -N \cdot f_{x_i x_j}.$$

Similarly,

$$f_{x_i} \cdot dN_p(f_{x_j}) = -N \cdot f_{x_i x_j}.$$

This porves that $dN_p(f_{x_i}) \cdot f_{x_j} = f_{x_i} \cdot dN_p(f_{x_j}) = -N \cdot f_{x_i x_j}$. □

The self-adjoint operator $-dN_p : TM_p \to TM_p$ is called the *shape operator* of the surface M at p. The symmetric bilinear form II_p associated to the self-adjoint map $-dN_p$ is called the *second fundamental form* of the surface, i.e.,

$$\mathrm{II}(f_{x_i}, f_{x_j}) = -dN(f_{x_i}) \cdot f_{x_j} = N \cdot f_{x_i x_j}.$$

We can write the second fundmental form II in terms of the dual basis dx_1, dx_2:

$$\mathrm{II} = \ell_{11}\ dx_1^2 + 2\ell_{12}\ dx_1 dx_2 + \ell_{22}\ dx_2^2,$$

where

$$\ell_{ij} = -N_{x_i} \cdot f_{x_j} = N \cdot f_{x_i x_j}.$$

We claim that the first and second fundamental forms are invariant under rigid motions. To see this, let $C(x) = Ax + v_0$ be a rigid motion of $\mathbb{R}^3$, i.e., A is an orthogonal transformation and v_0 is a constant vector in $\mathbb{R}^3$. Let $f : \mathcal{O} \to U \subset M$ is a local coordinate system of the embedded surface M in $\mathbb{R}^3$. Then $\hat{f} = C \circ f$ is a local coordinate system for the new surface $C(M)$, $\hat{f}_{x_i} = Af_{x_i}$, and $\hat{N} = AN$ is normal to $C(M)$. Since A is an orthogonal transformation,

$$(Av_1) \cdot (Av_2) = v_1 \cdot v_2.$$

Let $\hat{\mathrm{I}}$ and $\hat{\mathrm{II}}$ denote the fundamental forms of $\hat{M} = C(M)$. Then

$$\begin{aligned} \hat{g}_{ij} = \hat{\mathrm{I}}(\hat{f}_{x_i}, \hat{f}_{x_j}) &= \hat{f}_{x_i} \cdot \hat{f}_{x_j} = Af_{x_i} \cdot Af_{x_j} \\ &= f_{x_i} \cdot f_{x_j} = \mathrm{I}(f_{x_i}, f_{x_j}) = g_{ij}. \end{aligned}$$

So $\hat{\mathrm{I}} = \mathrm{I}$. Similarly, $\hat{\mathrm{II}} = \mathrm{II}$. This proves our claim.

Example 3.3.2. Fundamental forms of a graph

Let $u : \mathcal{O} \to \mathbb{R}$ be a smooth function, and $f : \mathcal{O} \to \mathbb{R}^3$ the graph of u, i.e., f is defined by

$$f(x_1, x_2) = (x_1, x_2, u(x_1, x_2)),$$

i.e., $M = f(\mathcal{O})$ is the graph of u. Then

$$f_{x_1} = (1, 0, u_{x_1}), \quad f_{x_2} = (0, 1, u_{x_2}),$$
$$f_{x_1x_1} = (0, 0, u_{x_1x_1}), \quad f_{x_1x_2} = (0, 0, u_{x_1x_2}), \quad f_{x_2x_2} = (0, 0, u_{x_2x_2}),$$
$$N = \frac{(-u_{x_1}, -u_{x_2}, 1)}{\sqrt{u_{x_1}^2 + u_{x_2}^2 + 1}}.$$

A direct computation gives

$$\begin{aligned}
g_{11} &= f_{x_1} \cdot f_{x_1} = 1 + u_{x_1}^2, \\
g_{12} &= f_{x_1} \cdot f_{x_2} = u_{x_1} u_{x_2}, \\
g_{22} &= f_{x_2} \cdot f_{x_2} = 1 + u_{x_2}^2, \\
h_{11} &= N \cdot f_{x_1x_1} = \frac{u_{x_1x_1}}{\sqrt{u_{x_1}^2 + u_{x_2}^2 + 1}}, \\
h_{12} &= N \cdot f_{x_1x_2} = \frac{u_{x_1x_2}}{\sqrt{u_{x_1}^2 + u_{x_2}^2 + 1}}, \\
h_{22} &= N \cdot f_{x_2x_2} = \frac{u_{x_2x_2}}{\sqrt{u_{x_1}^2 + u_{x_2}^2 + 1}}.
\end{aligned}$$

So the two fundamental forms for $f(\mathcal{O})$ are

$$\mathrm{I} = (1 + u_{x_1}^2) dx_1^2 + 2u_{x_1} u_{x_2} dx_1 dx_2 + (1 + u_{x_2}^2) dx_2^2,$$
$$\mathrm{II} = \frac{1}{\sqrt{1 + u_{x_1}^2 + u_{x_2}^2}} \; (u_{x_1x_1} dx_1^2 + 2u_{x_1x_2} dx_1 dx_2 + u_{x_2x_2} dx_2^2).$$

Proposition 3.3.3. *Locally a surface is a graph.*

Proof. Let $f = (f_1, f_2, f_3) : U \to \mathbb{R}^3$ be a surface. Given $p = (x_0, y_0) \in U$, since f_{x_1}, f_{x_2} are linearly independent,

$$\det \begin{pmatrix} i & j & k \\ (f_1)_{x_1} & (f_2)_{x_1} & (f_3)_{x_1} \\ (f_1)_{x_2} & (f_2)_{x_2} & (f_3)_{x_3} \end{pmatrix} \; (p) \; \neq 0.$$

So one of the following 2×2 determinants at p must be non-zero:

$$\begin{pmatrix} (f_1)_{x_1} & (f_2)_{x_1} \\ (f_1)_{x_2} & (f_2)_{x_2} \end{pmatrix}, \quad \begin{pmatrix} (f_1)_{x_1} & (f_3)_{x_1} \\ (f_1)_{x_2} & (f_3)_{x_2} \end{pmatrix}, \quad \begin{pmatrix} (f_2)_{x_1} & (f_3)_{x_1} \\ (f_2)_{x_2} & (f_3)_{x_2} \end{pmatrix}.$$

Suppose

$$\det \begin{pmatrix} (f_1)_{x_1} & (f_2)_{x_1} \\ (f_1)_{x_2} & (f_2)_{x_2} \end{pmatrix} (p) \; \neq 0.$$

Let $\pi_{12} : \mathbb{R}^3 \to \mathbb{R}^2$ denote the projection of $\mathbb{R}^3$ to the first two coordinates. Then the differential of the map $\pi_{12} \circ f : U \to \mathbb{R}^2$ at p is given by the matrix

$$\begin{pmatrix} (f_1)_{x_1} & (f_2)_{x_1} \\ (f_1)_{x_2} & (f_2)_{x_2} \end{pmatrix}$$

at p, which is non-singular. By the Inverse Function Theorem, there exist open subsets $\mathcal{O}_1, \mathcal{O}_2$ containing p and $(f_1(p), f_2(p))$ respectively so that $\pi_{12} \circ f$ maps $\mathcal{O}_1$ diffeomorphically to $\mathcal{O}_2$. Let $h : \mathcal{O}_2 \to \mathcal{O}_1$ denote the inverse of $\pi_{12} \circ f : \mathcal{O}_1 \to \mathcal{O}_2$, and $\tilde{f}(y_1, y_2) = f \circ h(y_1, y_2)$. In other words, we have changed coordinates (x_1, x_2) to (y_1, y_2) and expressed $(x_1, x_2) \in \mathcal{O}_1$ in terms of $(y_1, y_2) \in \mathcal{O}_2$. Hence $f_3(x_1, x_2) = f_3(h(y_1, y_2))$ is a function of y_1 and y_2. So we see that

$$\tilde{f}(y_1, y_2) = (y_1, y_2, f_3(h(y_1, y_2)))$$

is a graph of $u = f_3 \circ h$ and $\tilde{f}$ is a local parametrization of M. □

Angle and arc length

Since I_p gives an inner product on TM_p for all $p \in M$, we can use I to compute the arc length of curves on M and the angle between two curves on M:

(1) if $\alpha_1, \alpha_2 : (-\epsilon, \epsilon) \to M$ are two smooth curves such that $\alpha_1(0) = \alpha_2(0)$, then the angle between α_1 and α_2 is defined to be the angle between $\alpha_1'(0)$ and $\alpha_2'(0)$,

(2) the arc length of a curve on M given by $\alpha(t) = f(x_1(t), x_2(t))$ with $t \in [a, b]$ is

$$\begin{aligned} \int_a^b ||\alpha'(t)|| dt &= \int_a^b ||f_{x_1} x_1' + f_{x_2} x_2'|| dt \\ &= \int_a^b \left(\sum_{i,j=1}^2 (f_{x_i} \cdot f_{x_j}) x_i' x_j' \right)^{\frac{1}{2}} dt \\ &= \int_a^b \left(\sum_{i,j=1}^2 g_{ij}(x_1(t), x_2(t)) x_i'(t) x_j'(t) \right)^{\frac{1}{2}} dt. \end{aligned}$$

Recall that the area of the parallelgram spanned by v_1, v_2 in $\mathbb{R}^3$ is equal to

$$||v_1||\ ||v_1||\ |\cos\theta| = ||v_1 \times v_2|| = \left(\det \begin{pmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 \\ v_1 \cdot v_2 & v_2 \cdot v_2 \end{pmatrix} \right)^{\frac{1}{2}},$$

where θ is the angle between v_1 and v_2.

Proposition 3.3.4. *Let $f : \mathcal{O} \to U \subset M$ be a local coordinate system of an embedded surface, and $g_{ij} = f_{x_i} \cdots f_{x_j}$ the coefficients of the first fundamental form. Then the area of U is*

$$\text{Area}(U) = \int_{\mathcal{O}} ||f_{x_1} \cdot f_{x_2}||\ dx_1 dx_2 = \int_{\mathcal{O}} \sqrt{\det(g_{ij})}\ dx_1 dx_2.$$

The second fundamental form has a very simple geometric meaning:

Proposition 3.3.5. *Let v be a unit tangent vector of $M = f(\mathcal{O})$ at $p \in M = f(\mathcal{O})$, and α the intersection curve of M and the plane through p spanned by v and the normal vector N_p of M at p. Then $\text{II}(v, v)$ is the curvature of the plane curve α at p.*

Proof. Write $\alpha(s) = f(x_1(s), x_2(s))$ with arc length parameter. Then

$$\alpha''(s) = \sum_{ij=1}^{2} f_{x_i x_j} x_i' x_j'.$$

Since $N(p)$ is normal to M at p and $\alpha'(0) = v$, $N(p)$ is normal to $\alpha'(0)$. But the curvature of the plane curve α at $s = 0$ is $\alpha''(0) \cdot N(p)$, and

$$\alpha''(0) \cdot N = \sum_{i,j=1}^{2} (f_{x_i x_j} \cdot N) x_i' x_j' = \sum_{ij=1}^{2} h_{ij} x_i' x_j' = \text{II}(\alpha'(0), \alpha'(0)).$$

□

3.4. **The Gaussian curvature and mean curvature.**

Definition 3.4.1. The eigenvalues λ_1, λ_2 of the shape operator $-dN_p$ are called the *principal curvatures* of M at p, the unit eigenvectors of $-dN_p$ are called the *principal directions*, $\det(-dN_p)$ is called the *Gaussian curvature*, and the trace of dN_p is called the *mean curvature* of M at p.

So we have

$$K(p) = \det(-dN_p), \quad H = \frac{1}{2}\text{tr}(-dN_p).$$

If λ_1, λ_2 are eigenvalues of $-dN_p$, then

$$K = \lambda_1 \lambda_2, \quad H = \lambda_1 + \lambda_2.$$

Proposition 3.4.2. *Let $f : \mathcal{O} \to U$ be a local coordinate system of an embedded surface M in $\mathbb{R}^3$, and g_{ij}, ℓ_{ij} the coefficients of* I, II *on U respectively. Then*

$$K = \frac{\det(\ell_{ij})}{\det(g_{ij})} = \frac{\ell_{11}\ell_{22} - \ell_{12}^2}{g_{11}g_{22} - g_{12}^2},$$
$$H = \frac{g_{22}\ell_{11} - 2g_{12}\ell_{12} + g_{11}\ell_{22}}{g_{11}g_{22} - g_{12}^2}$$

Proof. By Proposition 3.1.4, $K = \det(G^{-1}L) = \frac{\det(G)}{\det(L)}$, and $H = \mathrm{tr}(G^{-1}L)$. The formula for H follows from

$$G^{-1} = \frac{1}{\det(g_{ij})} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix}.$$

□

Suppose v_1, v_2 are unit eigenvectors of $-dN_p$ with eigenvalues $\lambda_1 > \lambda_2$, then $\mathrm{II}_p(u,u)$ assumes minimum value λ_2 on the unit circle of TM_p at v_2 and maximum value λ_1 at v_1.

Below we compute the K and H of a plane, a sphere, a cylinder, and a graph.

Example 3.4.3. Let M be a plane. Let us parametrize it as a graph $f(x_1, x_2) = (x_1, x_2, a_1x_1 + a_2x_2)$. Then

$$f_{x_1} = (1, 0, a_1), \quad f_{x_2} = (0, 1, a_2), \quad N = \frac{(a_1, a_2, -1)}{\sqrt{a_1^2 + a_2^2 + 1}}.$$

Since N is a constant vector, $N_{x_i} = 0$. So the shape operator $dN_p = 0$. Hence $K = 0$ and $H = 0$. The first and second fundamental forms are

$$\mathrm{I} = (1 + a_1^2)dx_1^2 + (1 + a_2^2)dx_2^2, \quad \mathrm{II} = 0.$$

Example 3.4.4. Let M be the sphere of radius r. Let $f(x_1, x_2)$ be a parametrization of a piece of the sphere. Note that the unit normal N of M at $f(x_1, x_2)$ is $\frac{1}{r}\ f(x_1, x_2)$. Hence $dN(f_x) = \frac{1}{r} f_x$ and $dN(f_y) = \frac{1}{r} f_y$. This means that $dN_p = \frac{1}{r}\mathrm{Id}$. So $K = \frac{1}{r^2}$ and $H = -\frac{2}{r}$.

Example 3.4.5. Let M be the cylinder parametrized as

$$f(x, y) = (r\cos x, r\sin x, y).$$

Then

$$f_x = (-r\sin x, r\cos x, 0),$$
$$f_y = (0, 0, 1),$$
$$N = (\cos x, \sin x, 0).$$

So $N_x = \frac{1}{r} f_x$ and $N_y = 0$. This means that dN_p has two eigenvalues $\frac{1}{r}$ and 0. Hence $K = 0$ and $H = -\frac{1}{r}$. The first and second fundamental forms are

$$\mathrm{I} = r^2 dx_1^2 + dy^2, \quad \mathrm{II} = r dx^2.$$

Note that both a plane and a cylinder have Gaussian curvature 0.

Example 3.4.6. K and H for a graph

Let $f(x_1, x_2) = (x_1, x_2, u(x_1, x_2))$ denote the graph of a $u : \mathcal{O} \to \mathbb{R}$. We have computed the first and second fundamental form in Example 3.3.2. By Proposition 3.4.2, we have

$$K = \frac{u_{x_1x_1} u_{x_2x_2} - u_{x_1x_2}^2}{(1 + u_{x_1}^2 + u_{x_2}^2)^2}, \tag{3.4.1}$$

$$H = \frac{(1 + u_{x_1}^2) u_{x_2x_2} - 2u_{x_1} u_{x_2} u_{x_1x_2} + (1 + u_{x_2}^2) u_{x_1x_1}}{(1 + u_{x_1}^2 + u_{x_2}^2)^{\frac{3}{2}}}. \tag{3.4.2}$$

Definition 3.4.7. A point p in a surface $M \subset \mathbb{R}^3$ is called an *umbilic point* if the shape operator at p is equal to $\lambda \mathrm{Id}$ for some scalar λ, or equivalently, the two prinicipal curvatures are equal at p.

Definition 3.4.8. A unit tangent direction u of an embedded surface M in $\mathbb{R}^3$ is called *asymptotic* if $\mathrm{II}(u, u) = 0$.

Exercise 3.4.9. (1) If $K(p_0) < 0$, then there exist exactly two asymptotic directions.

(2) If p is not an umbilic point of M, then there exists a local smooth o.n. frame v_1, v_2 on M near p so that $v_1(q), v_2(q)$ are principal directions of M at q.

Next we give some geometric interpretation of the Gaussian curvature K. The first is a consequence of the Taylor expansion:

Proposition 3.4.10. *Let M be an embedded surface in $\mathbb{R}^3$, and $p \in M$.*

1. *If $K(p) > 0$, then there is a small neighborhood $\mathcal{O}_p$ in M of p such that $\mathcal{O}_p$ lies on one side of the tangent plane of M at p.*
2. *If $K(p) < 0$, then given any small open subset of M containing p, M lies on both sides of the tangent plane of M at p.*

Proof. After rotation and translation, we may assume that $p = (0, 0, 0)$, M is parametrized as a graph

$$f(x_1, x_2) = (x_1, x_2, u(x_1, x_2))$$

in a neighborhood of p, and the unit normal $N_p = (0,0,1)$. This means that we assume

$$u(0,0) = 0, \quad u_{x_i}(0,0) = 0.$$

Assume $K(p) > 0$, then $\lambda_1(p)\lambda_2(p) > 0$. So either both λ_1, λ_2 are positive or both are negative. So II$(0,0)$ is definite. Use Example 3.3.2 formula (3.4.1) to conclude that $(u_{x_ix_j}(0,0))$ is positive definite or negative definite. Suppose it is positive definite. Then the quadratic form $\sum_{ij} u_{x_ix_j}(0,0)y_iy_j > 0$ for all $y \in \mathbb{R}^2$. On the other hand, the Taylor theorem of u implies that

$$\begin{aligned} &u(x_1,x_2)\\ &= u(0,0) + u_{x_1}(0,0)x_1 + u_{x_2}(0,0)x_2 + \frac{1}{2}\sum_{ij} u_{x_ix_j}(0,0)x_ix_j + o(r^2),\\ &= \frac{1}{2}\sum_{ij} u_{x_ix_j}(0,0)x_ix_j + o(r^2) \end{aligned}$$

where $r^2 = \sum_i x_i^2$. So there exists $\delta > 0$ so that if $0 < \|\, x \,\| < \delta$, then $u(x_1,x_2) > 0$. This proves (1). Statement (2) can be proved in a similar manner. □

Proposition 3.4.11. *Let $f : \mathcal{O} \to U \subset M$ be a local coordinate system of an embedded surface M in $\mathbb{R}^3$, N the unit normal field, and $p = f(q) \in f(\mathcal{O})$. Then*

$$K(p) = \lim_{\Omega \to q} \frac{Area(N(\Omega))}{Area(f(\Omega))}.$$

Proof. By Proposition 3.3.4 we have

$$\begin{aligned} \text{Area}(f(\Omega)) &= \int_W \|f_x \times f_y\| dxdy,\\ \text{Area}(N(\Omega)) &= \int_W \|N_x \times N_y\| dxdy. \end{aligned}$$

Since $dN(f_x) = N_x$ and $dN(f_y) = N_y$, we have

$$\|N_x \times N_y\| = |\det(dN)| \|f_x \times f_y\| = |K| \|f_x \times f_y\|.$$

Proposition follows from the fact that the Gaussian curvature K is a smooth function. □

To understand the meaning of mean curvature H we need calculus of variations.

4. **Calculus of variations**

Let Ω be an open subset of $\mathbb{R}^n$ with smooth boundary such that the closure $\bar{\Omega}$ is compact, and $\partial\Omega$ the boundary of Ω. Let $\alpha : \partial\Omega \to \mathbb{R}^m$ be a smooth map, and $C^\infty_\alpha(\Omega, \mathbb{R}^m)$ the space of smooth maps u from Ω to $\mathbb{R}^m$ so that $u|\partial\Omega = \alpha$. Let $L : \Omega \times (\mathbb{R}^m)^{n+1} \to \mathbb{R}$ be a smooth map, and $J : C^\infty_\alpha(\Omega, \mathbb{R}) \to \mathbb{R}$ the function defined by

$$J(u) = \int_\Omega L(x, u, u_{x_1}, \ldots, u_{x_n})\ dx_1 \cdots dx_n.$$

The function L is called a *Lagrangian*, and J the functional defined by L. The calculus of variations studies the condition on u when u is a "critical point" of L. Since $C^\infty_\alpha(\Omega, \mathbb{R}^m)$ is an infinite dimensional vector space, we need to extend the concept of critical point to this function space.

Recall that the following statements are equivalent for a smooth function $f : \mathbb{R}^n \to \mathbb{R}$:

(1) p is a critical point of f,
(2) $f_{x_1}(p) = \cdots = f_{x_n}(p) = 0$,
(3) $\frac{d}{dt}\Big|_{t=0} f(\alpha(t)) = 0$ for all smooth curve α through p.

We will use (3) to define critical points for the functional J defined by the Lagrangian L. Let $u \in C^\infty_\alpha(\Omega, \mathbb{R}^m)$.

Definition 4.0.12. A smooth *variation* of $u \in C^\infty_\alpha(\Omega, \mathbb{R}^m)$ is a curve $\beta : (-\epsilon, \epsilon) \to C^\infty_\alpha(\Omega, \mathbb{R}^m)$ so that $\beta(0) = u$ and $(s, x) \mapsto \beta(s)(x)$ is smooth. Note that $v(x) = \frac{\partial}{\partial s}\big|_{s=0} \beta(s)(x)$ is a smooth map from Ω to $\mathbb{R}^m$ that vanishes on $\partial\Omega$.

Definition 4.0.13. $u \in C^\infty_\alpha(\Omega, \mathbb{R}^m)$ is a *critical point* of J if

$$(J \circ \beta)'(0) = 0$$

for all smooth variations β of u in $C^\infty_\alpha(\Omega, \mathbb{R}^m)$.

Proposition 4.0.14. *If $u \in C^\infty_\alpha(\Omega, \mathbb{R}^m)$ assumes maximum (or minimum) of J, then u is a critical point of J.*

Proof. Let $\beta : (-\epsilon, \epsilon) \to C^\infty_\alpha(\Omega, \mathbb{R}^m)$ be a smooth variation of u in $C^\infty_\alpha(\Omega, \mathbb{R}^m)$. Then $s = 0$ is an exteme point of $J \circ \beta : (-\epsilon, \epsilon) \to \mathbb{R}$ and $\frac{d}{ds}|_{s=0}(J \circ \beta) = 0$. This implies that extreme point of J is a critical point of J. □

To derive the condition for u to be a critical point of the functional J, we need a Lemma:

Lemma 4.0.15. *Given a smooth map $(g_1, \cdots, g_m) : \Omega \to \mathbb{R}^m$, if*

$$\int_\Omega \sum_{j=1}^n g_j(x)h_j(x)dx_1 \cdots dx_n = 0$$

for all smooth $(h_1, \cdots, h_n) : \Omega \to \mathbb{R}^n$ with $h \mid \partial\Omega = 0$, then $g_j \equiv 0$ for all $1 \le j \le n$.

Proof. We will prove this Lemma when $m = 1$ and $n = 1$. For general m, n, we use the Stoke's Theorem instead of the Fundamental theorem of calculus. Since in this notes, we only need to deal with the case when $n = 1$ or 2, we will give a proof when $n = 2$ in more detail later.

If g is not identically zero, then we may assume there is an interval $(\delta_1, \delta_2) \subset [0, 1]$ so that g is positive on (δ_1, δ_2). Choose $\epsilon > 0$ so that $\epsilon < \frac{(\delta_2 - \delta_1)}{2}$. Now choose h to be a non-negative function, which vanishes outside (δ_1, δ_2) and is $h > 0$ in $(\delta_1 + \epsilon, \delta_2 - \epsilon)$. Then $gh > 0$ on $(\delta_1 + \epsilon, \delta_2 - \epsilon)$ and is non-negative on $[0, 1]$. So $\int_0^1 g(t)h(t) > 0$, a contradiction. Hence $g \equiv 0$. □

We will derive the condition for critical points of L in the later sections.

4.1. Calculus of Variations of one variable.

In this section, we derive the condition for a curve on an embedded surface M in $\mathbb{R}^3$ joining $p, q \in M$ that has shortest arc length.

Given $p, q \in \mathbb{R}^n$, let $\mathcal{M}_{p,q}$ denote the space of smooth curves $x : [0, 1] \to \mathbb{R}^n$ so that $x(0) = p$ and $x(1) = q$. Let $L : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a smooth function, and $J : \mathcal{M}_{p,q} \to \mathbb{R}$ the functional defined by

$$J(x) = \int_0^1 L(t, x(t), x'(t))dt. \tag{4.1.1}$$

Suppose γ is a critical point of J. Let β be a variation of γ, and $\beta'(0) = h$. Let

$$b(s, t) = \beta(s)(t).$$

We have

$$b(0, t) = \gamma(t), \quad h(t) = \frac{\partial b}{\partial s}(0, t), \quad h(0) = h(1) = 0.$$

By definition of critical point and integration by part, we get

$$
\begin{aligned}
0 &= \frac{d}{ds}\bigg|_{s=0} J(\beta(s)) = \frac{d}{ds}\bigg|_{s=0} \int_0^1 L\left(t, b(s,t), \frac{\partial b}{\partial t}\right)\ dt \\
&= \int_0^1 \sum_j \frac{\partial L}{\partial x_j}\ \frac{\partial b_j}{\partial s} + \frac{\partial L}{\partial y_j}\ \frac{\partial^2 b}{\partial t \partial s}\bigg|_{s=0}\ dt \\
&= \int_0^1 \sum_j \left(\frac{\partial L}{\partial x_j} h_j + \sum_j \frac{\partial L}{\partial y_j} h_j'\right)\ dt, \quad \text{here } ' \text{ means } \frac{\mathrm{d}}{\mathrm{dt}}, \\
&= \int_0^1 \sum_j \frac{\partial L}{\partial x_j} h_j dt + \left(\sum_j \frac{\partial L}{\partial y_j} h_j\right)\bigg|_0^1 - \int_0^1 \left(\sum_j \frac{\partial L}{\partial y_j}\right)' h_j\ dt \\
&= \int_0^1 \sum_j \left(\frac{\partial L}{\partial x_j}(t, \gamma(t), \gamma'(t)) - \left(\frac{\partial L}{\partial y_j}(t, \gamma(t), \gamma'(t))\right)'\right) h_j\ dt \\
&\qquad + \sum_j \frac{\partial L}{\partial y_j} h_j\bigg|_0^1.
\end{aligned}
$$

Since $h(0) = h(1) = 0$, the boundary term is zero. Hence we have

$$\int_0^1 \sum_j \left(\frac{\partial L}{\partial x_j} - \left(\frac{\partial L}{\partial y_j}\right)'\right) h_j\ dt = 0$$

for all smooth $h : [0,1] \to \mathbb{R}^n$ with $h(0) = h(1) = 0$. By Lemma 4.0.15, we get

$$\frac{\partial L}{\partial x_j}(t, \gamma(t), \gamma'(t)) - \left(\frac{\partial L}{\partial y_j}(t, \gamma(t), \gamma'(t))\right)' = 0.$$

So we have proved

Theorem 4.1.1. *Let $J : \mathcal{M}_{p,q} \to \mathbb{R}$ be the functional defined by $L(t,x,y)$, i.e., $J(\gamma) = \int_0^1 L(t, \gamma(t), \gamma'(t))dt$. Then $\gamma \in \mathcal{M}_{p,q}$ is a critical point of J if and only if*

$$\frac{\partial L}{\partial x_j}(t, \gamma, \gamma') = \left(\frac{\partial L}{\partial y_j}(t, \gamma, \gamma')\right)'. \tag{4.1.2}$$

Equation (4.1.2) is called the *Euler-Lagrange* equation for the functional J.

Example 4.1.2. Let $L(t,x,y) = \sum_{i=1}^n y_i^2$, and $E : \mathcal{M}_{p,q} \to \mathbb{R}$ the functional defined by L, i.e.,

$$E(\gamma) = \int_0^1 \sum_{i=1}^n (\gamma_i'(t))^2 dt.$$

The functional E is called the *energy functional*. A direct computation implies that Euler-Lagrangian equation for E is $y''(t) = 0$. So $y(t)$ is linear. Since $y(0) = p$ and $y(1) = q$, we have $y(t) = p + t(q - p)$ a straight line.

4.2. **Geodesics.**

Let M be an embedded surface in $\mathbb{R}^3$, and $f : \mathcal{O} \to U \subset M$ a coordinate system. A smooth curve $\alpha : [0,1] \to U \subset M$ is given by a smooth curve $(x_1(t), x_2(t))$ in $\mathcal{O}$ such that

$$\alpha(t) = f(x_1(t), x_2(t)).$$

So $\alpha'(t) = \sum_i f_{x_i} x_i'$. Since the arc length of α is approximated by

$$\sum \|\alpha'(t)\| \triangle t,$$

the arc length of α is given $\int_0^1 \|\alpha'(t)\| dt$. But

$$\begin{aligned}\|\alpha'\|^2 = \alpha' \cdot \alpha' &= \left(\sum_i f_{x_i} x_i'\right) \cdot \left(\sum_j f_{x_j} x_j'\right) = \sum_{ij} f_{x_i} \cdot f_{x_j} x_i' x_j' \\ &= \sum_{ij} g_{ij}(x) x_i' x_j' = \mathrm{I}(\alpha', \alpha').\end{aligned}$$

So the arc length of α is

$$\mathcal{L}(x) = \int_0^1 \sqrt{\sum_{ij} g_{ij}(x) x_i' x_j'}\ dt.$$

We want to calculate the Euler Lagrange equation for the energy and arc length functionals on the surface $M = f(\mathcal{O})$ in $\mathbb{R}^3$. Let $p = (x_1^0, x_2^0)$, $q = (x_1^1, x_2^1)$, and $\mathcal{M}_{p,q}$ the space of all smooth $x : [0,1] \to \mathcal{O}$ such that $x(0) = p$ and $x(1) = q$. Suppose the first fundamental form of M is $\mathrm{I} = \sum_{ij} g_{ij}(x) dx_i dx_j$. If $x : [0,1] \to \mathcal{O}$ lies in $\mathcal{M}_{p,q}$, then $\gamma = f \circ x$ is a curve on M joining $f(p)$ to $f(q)$ and

$$||\gamma'(t)||^2 = \sum_{ij} g_{ij}(x(t)) x_i'(t) x_j'(t).$$

Let

$$L(x, y) = \frac{1}{2} \sum_{ij} g_{ij}(x) y_i y_j, \tag{4.2.1}$$

and E and $\mathcal{L}$ denote *the energy and arc length functionals*

$$E(x) = \int_0^1 L(x(t), x'(t))dt,$$
$$\mathcal{L}(x) = \int_0^1 \sqrt{L(x(t), x'(t))}dt.$$

The Euler-Lagrange equation for the functional E and $\mathcal{L}$ are

$$\frac{\partial L}{\partial x_k} = \left(\frac{\partial L}{\partial x'_k}\right)'. \qquad k = 1, 2, \tag{4.2.2}$$

$$\frac{\frac{\partial L}{\partial x_k}}{2\sqrt{L}} = \left(\frac{\frac{\partial L}{\partial x'_k}}{2\sqrt{L}}\right)'. \tag{4.2.3}$$

respectively.

A direct computation gives

$$\frac{\partial L}{\partial x_k} = \frac{1}{2}\sum_{ij} g_{ij,k}x'_i x'_j, \qquad \text{where } g_{ij,k} = \frac{\partial g_{ij}}{\partial x_k},$$
$$= \left(\frac{\partial L}{\partial x'_k}\right)' = \left(\sum_i g_{ik}x'_i\right)' = \sum_{i,m} g_{ik,m}x'_i x'_m + \sum_i g_{ik}x''_i,$$

Playing with indices to get

$$\sum_{i,m} g_{ik,m}x'_i x'_m = \sum_{ij} g_{ik,j}x'_i x'_j = \sum_{ji} g_{jk,i}x'_j x'_i.$$

Note that the first equality is true because we just let $m = j$, the second equality is true because we interchange i, j in the summand. Hence

$$\sum_{i,m} g_{ik,m}x'_i x'_m = \frac{1}{2}\left(\sum_{ij} g_{ik,j}x'_i x'_j + \sum_{ji} g_{jk,i}x'_j x'_i\right).$$

This implies that the Euler-Lagrange equation of the energy functional E is

$$\sum_i g_{ik}x''_i + \frac{1}{2}\sum_{ij}\left(g_{ki,j} + g_{jk,i} - g_{ij,k}\right)x'_i x'_j = 0. \tag{4.2.4}$$

Let g^{ij} denote the ij-th entry of the inverse of the matrix of (g_{ij}). Multiply (4.2.4) by g^{km} and sum over k to get

$$\sum_{i,k} g^{km}g_{ik}x''_i + \frac{1}{2}\sum_{ijk} g^{km}\left(g_{ki,j} + g_{jk,i} - g_{ij,k}\right)x'_i x'_j = 0.$$

But $\sum_k g^{ik} g_{km} = \delta_{im}$ and (g_{ij}) is symmetric. So we get

$$x_m'' + \frac{1}{2} \sum_{ijk} g^{km} \left(g_{ki,j} + g_{jk,i} - g_{ij,k} \right) x_i' x_j' = 0. \tag{4.2.5}$$

So we have proved the first part of the following theorem:

Theorem 4.2.1. *If $x(t)$ is a critical point of the energy functional E, then*

(1) *x is a solution of*

$$x_m'' + \sum_{ij} \Gamma_{ij}^m x_i' x_j' = 0, \tag{4.2.6}$$

where $\Gamma_{ij}^m = \frac{1}{2} \sum_k g^{km} (g_{ki,j} + g_{jk,i} - g_{ij,k})$.

(2) *$\|x'(t)\|$ is independent of t.*

Proof. It remains to prove (2). We compute as follows:

$$\begin{aligned}
\left(\sum_{ij} g_{ij}(x) x_i' x_j' \right)' &= \sum_{ijk} g_{ij,k} x_i' x_j' x_k' + \sum_{ij} g_{ij} x_i'' x_j' + g_{ij} x_i' x_j'' \\
&= \sum_{ijk} g_{ij,k} x_i' x_j' x_k' - \left(\sum_{ijm} g_{ij} \Gamma_{mk}^i x_m' x_k' x_j' + g_{ij} x_i' \Gamma_{km}^j x_k' x_m' \right) \\
&= \sum_{ijk} g_{ij,k} x_i' x_j' x_k' - \sum_{ijm} g_{mj} \Gamma_{ik}^m x_i' x_k' x_j' - \sum_{ijm} g_{im} x_i' \Gamma_{kj}^m x_k' x_j' \\
&= \sum_{ijk} g_{ij,k} x_i' x_j' x_k' - \frac{1}{2} \sum_{ijkmr} g_{mj} g^{mr} [ik, r] x_i' x_k' x_j' - \frac{1}{2} g_{im} g^{mr} [kj, r] x_i' x_k' x_j' \\
&= \sum_{ijk} \left(g_{ij,k} - \frac{1}{2} \sum_r (\delta_{jr} [ik, r] + \delta_{ir} [kj, r]) \right) x_i' x_k' x_j' \\
&= \sum_{ijk} \left(g_{ij,k} - \frac{1}{2} [ik, j] - \frac{1}{2} [kj, i] \right) x_i' x_j' x_k' \\
&= \sum_{ijk} \left(g_{ij,k} - \frac{1}{2} (g_{kj,i} + g_{ij,k} - g_{ik,j}) - \frac{1}{2} (g_{ki,j} + g_{ij,k} - g_{kj,i}) \right) x_i' x_j' x_k' \\
&= 0,
\end{aligned}$$

where $[ij, k] = g_{ik,j} + g_{jk,i} - g_{ij,k}$. (In the above computation, the second term of the third line is obtained by interchanging i and m, and the third term is obtained by interchanging j and m). This proves $\|x'(t)\|^2 = \sum_{ij} g_{ij}(x(t)) x_i'(t) x_j'(t)$ is constant. □

Proposition 4.2.2. *If $k : [a, b] \to [0, 1]$ is a diffeomorphism, then*

(1) $\mathcal{L}(x \circ k) = \mathcal{L}(x)$, *i.e, the arc length of a curve on M does not depend on the parametrization of the curve.*
(2) *if x is a critical point of J, then so is $x \circ k$.*

Proof. Suppose $s = k(r)$ and $y(r) = x(k(r))$. Then $\frac{dy}{dr} = \frac{dx}{ds}\frac{ds}{dr}$. Both statements follow from the change of variable formula in integration. □

Proposition 4.2.3. *If x is a critical point of the arc length functional $\mathcal{L}$ and is parametrized proportional to its arc length, then x is a critical point of the energy functional E. Conversely, if x is a critical point of E, then x is parametrized proportional to its arc length and is a critical point of $\mathcal{L}$.*

Proof. If x is a critical point of $\mathcal{L}$, then we may assume that $\alpha(t) = f(x_1(t), x_2(t))$ is parametrized by arc length parameter. This means we may assume $\|\alpha'(t)\| \equiv 1$. But

$$\|\alpha'(t)\|^2 = \sum_{ij} g_{ij} x_i' x_j' = L(x, x') \equiv 1.$$

Since x is a critical point of $\mathcal{L}$, x is a solution of (4.2.3). But $L(x, x') \equiv 1$ implies that x is a solution of (4.2.2), i.e., x is a critical point for E.

Conversely, if x is a critical point of E, then by Theorem 4.2.1 (ii), $L(x, x')$ is constant. If $L(x(t), x'(t))$ is constant, then (4.2.3) and(4.2.2) are the same. This proves that if x is a critical point of E, then x is a critical point of $\mathcal{L}$. □

A critical point of the arch length functional $\mathcal{L}$ on $\mathcal{M}_{p,q}$ is called a *geodesic*. Equation (4.2.6) is called the *geodesic equation*.

It follows from Proposition 4.0.14 that if $\gamma(t) = f(x_1(t), x_2(t))$ is a curve on M joining p, q, which is parametrized by arc length, and has shortest arc length, then γ is a geodesic and $(x_1(t), x_2(t))$ is a solution of the geodesic equation.

4.3. Calculus of variations of two variables.

Suppose Ω is an open subset of $\mathbb{R}^2$ such that the boundary $\partial\Omega$ is a simple closed curve. Fix a function $\gamma : \partial\Omega \to \mathbb{R}$, let $\mathcal{C}_\gamma(\Omega, \mathbb{R})$ denote the set of all smooth maps $u : \Omega \to \mathbb{R}$ so that $u \mid \partial\Omega = \gamma$. Let

$$L : \Omega \times (\mathbb{R}^3)^3 \to \mathbb{R}$$

be a smooth function, and $J : \mathcal{C}_\gamma(\Omega, \mathbb{R}) \to \mathbb{R}$ the functional defined by

$$J(u) = \int_\Omega L(x, y, u(x, y), u_x(x, y), u_y(x, y)) dxdy. \tag{4.3.1}$$

We want to derive the condition for u being a critical point of J. This can be carried out in a similar manner as for one variable case except that we need to use the two dimensional version of The Fundamental Theorem of Calculus, i.e., the Green's Formula or the Stoke's theorem for dimension 2.

First, we recall the definition of the line integral,

$$\oint_{\partial\Omega} P(x,y)dx + Q(x,y)dy.$$

Let $(x,y) : [a,b] \to \mathbb{R}^2$ be a parametrization of the boundary curve $\partial\Omega$. Then

$$\oint_{\partial\Omega} Pdx + Qdy = \int_a^b P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)\ dt.$$

The Green's formula is

Theorem 4.3.1. *Let $P, Q : \Omega \to \mathbb{R}$ be smooth functions. Then*

$$\oint_{\partial\Omega} P(x,y)dx + Q(x,y)dy = \int\int_{\Omega} (-P_y + Q_x)dxdy. \tag{4.3.2}$$

Suppose u is a critcal point of J. We want to find the condition on u. Let β be a variation of u in $\mathcal{C}_\gamma(\Omega,\mathbb{R})$, and $h = \beta'(0)$, i.e., $h(x,y) = \frac{\partial}{\partial s}|_{s=0}\beta(s)(x,y)$. Since $\beta(s)|\mid\partial\Omega = \gamma$, we have $h|\partial\Omega = 0$. Compute directly to get

$$\begin{aligned}
&\frac{d}{ds}\Big|_{s=0} J(\beta(s)) = \int_\Omega \frac{\partial L}{\partial u}h + \frac{\partial L}{\partial u_x}h_x + \frac{\partial L}{\partial u_y}h_y\ dxdy \\
&= \int_\Omega \frac{\partial L}{\partial u}h + \left(\frac{\partial L}{\partial u_x}h\right)_x - \left(\frac{\partial L}{\partial u_x}\right)_x h + \left(\frac{\partial L}{\partial u_y}h\right)_y - \left(\frac{\partial L}{\partial u_y}\right)_y h\ dxdy \\
&= \int_\Omega \left(\frac{\partial L}{\partial u} - \left(\frac{\partial L}{\partial u_x}\right)_x - \left(\frac{\partial L}{\partial u_y}\right)_y\right) h\ dxdy \\
&+ \quad \int_\Omega \left(\frac{\partial L}{\partial u_x}h\right)_x + \left(\frac{\partial L}{\partial u_y}h\right)_y dxdy.
\end{aligned}$$

It follows from the Green's formula (4.3.2) that the second term is

$$\int_{\partial\Omega} \frac{\partial L}{\partial u_x}\ h\ dy - \frac{\partial L}{\partial u_y}\ h\ dx.$$

But $h \mid \partial\Omega = 0$. So the line integral is zero. This shows that if u is a critical point of J, then

$$\int_\Omega \left(\frac{\partial L}{\partial u} - \left(\frac{\partial L}{\partial u_x}\right)_x - \left(\frac{\partial L}{\partial u_y}\right)_y\right) h\ dxdy = 0$$

for all smooth functions $h : \Omega \to \mathbb{R}$ that vanishes on $\partial\Omega$.

A similar proof gives the following analogue of Lemma 4.0.15 for $\mathbb{R}^n$.

Lemma 4.3.2. *Let $\Omega \subset \mathbb{R}^n$ be an open subset, and $f : \bar{\Omega} \to \mathbb{R}$ a smooth map. If*

$$\int_\Omega f(x)g(x)\ dx_1 \cdots dx_n = 0$$

for all smooth $g : \bar{\Omega} \to \mathbb{R}$ that vanishes on $\partial\Omega$, then $f = 0$.

Hence we have proved

Theorem 4.3.3. *If u is a critical point of J defined by* (4.3.1)*, then*

$$\frac{\partial L}{\partial u} - \left(\frac{\partial L}{\partial u_x}\right)_x - \left(\frac{\partial L}{\partial u_y}\right)_y = 0. \tag{4.3.3}$$

(This is the Euler-Lagrange equation of J defined by (4.3.1)*).*

Example 4.3.4. Let $\Omega \subset \mathbb{R}^2$, $\alpha : \partial\Omega \to \mathbb{R}$ a smooth function, and $L(u) = u_x^2 + u_y^2$ for $u \in C^\infty_\gamma(\Omega, \mathbb{R})$. Use Theorem 4.3.3 to compute directly to see that u is a critical point of the functional J defined by L if and only if

$$u_{xx} + u_{yy} = 0.$$

4.4. **Minimal surfaces.**

Let M be an embedded surface in $\mathbb{R}^3$, and $f : \mathcal{O} \to U$ a local coordinate system on M, and $\bar{\Omega} \subset \mathcal{O}$ a compact domain. By Proposition 3.3.4, the area of the parallelgram spanned by f_{x_1}, f_{x_2} is $\sqrt{\det(g_{ij})}$. To approximate the area of $f(\Omega)$, we first divide the domain Ω into small squares, and approximate the area of the image of f of each square by the area of the parallelgram spanned by $f_{x_1}\triangle x_1$ and $f_{x_2}\triangle x_2$, then we sum up these approximations

$$\sum \sqrt{\det(g_{ij})}\ \triangle x_1 \triangle x_2.$$

So the area of $f(\Omega)$ is

$$\text{area} = \int\int_\Omega \sqrt{\det(g_{ij}(x))}\ dx_1 dx_2 = \iint_\Omega \sqrt{g_{11}g_{22} - g_{12}^2}\ dx_1 dx_2.$$

We want to show that if $f(\Omega)$ has minimum area among all surfaces in $\mathbb{R}^3$ with fixed boundary $f(\partial\Omega)$, then the mean curvature of M must be zero. Since the condition for critical point of a variational functional is computed locally and locally a surface in $\mathbb{R}^3$ is a graph, we may assume that $f : \Omega \to U$ is a graph and the variations are done through graphs.

A surface in $\mathbb{R}^3$ is called *minimal* if its mean curvature $H = 0$. We will show that if a surface M in $\mathbb{R}^3$ is a critical point of the area functional then the mean cruvature of M must be zero, i.e., M is minimal. In fact, we will prove that the Euler-Lagrangian equation for the area functional is the equation $H = 0$.

Let $u : \mathcal{O} \to \mathbb{R}$, and $f(x_1, x_2) = (x_1, x_2, u(x_1, x_2))$ be the graph of u. Let $\bar{\Omega} \subset \mathcal{O}$. If $M = f(\Omega)$ has minimum area among all surfaces in $\mathbb{R}^3$ that have the same boundary as $f(\Omega)$, then u must be a critical point of the area functional

$$\mathcal{A}(u) = \int\int_\Omega \sqrt{\det(g_{ij})}\, dx_1 dx_2.$$

It is computed in Example 3.3.2 that

$$g_{ii} = 1 + u_{x_i}^2, \quad g_{12} = u_{x_1} u_{x_2}.$$

So $\det(g_{ij}) = (1 + u_{x_1}^2 + u_{x_2}^2)$. The area functional is

$$\mathcal{A}(u) = \int\int_\Omega (1 + u_{x_1}^2 + u_{x_2}^2)^{\frac{1}{2}} dx_1 dx_2.$$

By Theorem 4.3.3, u must satisfy the Euler-Lagrangian equation (4.3.3). A direct computation gives

$$\left((1 + u_{x_1}^2 + u_{x_2}^2)^{-\frac{1}{2}}\ u_{x_1}\right)_{x_1} + \left((1 + u_{x_1}^2 + u_{x_2}^2)^{-\frac{1}{2}}\ u_{x_2}\right)_{x_2} = 0.$$

So we have

$$\frac{(1 + u_{x_2}^2) u_{x_1 x_1} - 2 u_{x_1} u_{x_2} u_{x_1 x_2} + (1 + u_{x_1}^2) u_{x_2 x_2}}{(1 + u_{x_1}^2 + u_{x_2}^2)^{\frac{3}{2}}} = 0.$$

But the left hand side is the mean curvature H of the graph of u (cf. Example 3.4.6 formula (3.4.2)). So the mean curvature must be zero, i.e., M is minimal. We have proved

Theorem 4.4.1. *The Euler-Lagrangian equation of the area functional $\mathcal{A}$ is $H = 0$.*

5. Fundamental Theorem of surfaces in $\mathbb{R}^3$

Let M be an embedded surface in $\mathbb{R}^3$, and $f : \mathcal{O} \to U \subset \mathbb{R}^3$ a local coordinate system on M. Our experience in curve theory tells us that we should find a moving frame on the surface and then differentiate the moving frame to get relations among the invariants. However, unlike the curves, there do not have natural local orthonormal frames on general surfaces in $\mathbb{R}^3$. We will use several different moving frames $F = (v_1, v_2, v_3)$ on the surface to derive the relations among local invariants. Express the x and y derivatives of the local frame v_i in terms of v_1, v_2, v_3, then their coefficients can be written in terms of the two fundamental forms. Since $(v_i)_{xy} = (v_i)_{yx}$, we obtain a PDE relation for I and II. This is the Gauss-Codazzi equation of the surface. Conversely, given two symmetric bilinear forms g, b on an open subset $\mathcal{O}$ of $\mathbb{R}^2$ such that g is positive definite and g, b satisfies the Gauss-Codazzi equation, then by the Frobenius Theorem there exists a unique surface in $\mathbb{R}^3$ having g, b as the first and second fundamental forms respectively.

5.1. **Gauss-Codazzi equations in arbitrary local coordinates.**

Let $f : \mathcal{O} \to U \subset M$ be a local coordinate system of an embedded surface M in $\mathbb{R}^3$. We use the frame (f_{x_1}, f_{x_2}, N), where

$$N = \frac{f_{x_1} \times f_{x_2}}{||f_{x_1} \times f_{x_2}||}$$

is the unit normal vector field. Since f_{x_1}, f_{x_2}, N form a basis of $\mathbb{R}^3$, the partial derivatives of f_{x_i} and N can be written as linear combinations of f_{x_1}, f_{x_2} and N. So we have

$$\begin{cases} (f_{x_1}, f_{x_2}, N)_{x_1} = (f_{x_1}, f_{x_2}, N)P, \\ (f_{x_1}, f_{x_2}, N)_{x_2} = (f_{x_1}, f_{x_2}, N)Q, \end{cases} \tag{5.1.1}$$

where $P = (p_{ij}), Q = (q_{ij})$ are $gl(3)$-valued maps. This means that

$$\begin{cases} f_{x_1x_1} = p_{11}f_{x_1} + p_{21}f_{x_2} + p_{31}N, \\ f_{x_2x_1} = p_{12}f_{x_1} + p_{22}f_{x_2} + p_{32}N, \\ N_{x_1} = p_{13}f_{x_1} + p_{23}f_{x_3} + p_{33}N, \end{cases} \qquad \begin{cases} f_{x_1x_2} = q_{11}f_{x_1} + q_{21}f_{x_2} + q_{31}N, \\ f_{x_2x_2} = q_{12}f_{x_1} + q_{22}f_{x_2} + q_{32}N, \\ N_{x_2} = q_{13}f_{x_1} + q_{23}f_{x_3} + q_{33}N. \end{cases}$$

Recall that the fundamental forms are given by

$$g_{ij} = f_{x_i} \cdot f_{x_j}, \quad \ell_{ij} = -f_{x_i} \cdot N_{x_j} = f_{x_ix_j} \cdot N.$$

We want to express P and Q in terms of g_{ij} and h_{ij}. To do this, we need the following Propositions.

Proposition 5.1.1. *Let V be a vector space with an inner product $(\, ,)$, $v_1, \cdots, v_n$ a basis of V, and $g_{ij} = (v_i, v_j)$. Let $\xi \in V$, $\xi_i = (\xi, v_i)$, and $\xi = \sum_{i=1}^n x_i v_i$. Then*

$$\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = G^{-1} \begin{pmatrix} \xi_1 \\ \cdot \\ \cdot \\ \xi_n \end{pmatrix},$$

where $G = (g_{ij})$.

Proof. Note that

$$\xi_i = (\xi, v_i) = (\sum_{j=1}^n x_j v_j, v_i) = \sum_{j=1}^n x_j (v_j, v_i) = \sum_{j=1}^n g_{ji} x_j.$$

So $(\xi_1, \cdots, \xi_n)^t = G(x_1, \cdots, x_n)^t$. □

Proposition 5.1.2. *The following statements are true:*

1. *The $gl(3)$ valued functions $P = (p_{ij})$ and $Q = (q_{ij})$ in equation (5.1.1) can be written in terms of g_{ij}, ℓ_{ij}, and first partial derivatives of g_{ij}.*
2. *The entries $\{p_{ij}, q_{ij} \mid 1 \le i, j \le 2\}$ can be computed from the first fundamental form.*

Proof. We claim that

$$f_{x_i x_j} \cdot f_{x_k}, \quad f_{x_i x_j} \cdot N, \quad N_{x_i} \cdot f_{x_j}, \quad N_{x_i} \cdot N,$$

can be expressed in terms of g_{ij}, ℓ_{ij} and first partial derivatives of g_{ij}. Then the Proposition follows from Proposition 5.1.1. To prove the claim, we proceed as follows:

(5.1.2)

$$\begin{cases} f_{x_i x_i} \cdot f_{x_i} = \frac{1}{2}(g_{ii})_{x_i}, & \\ f_{x_i x_j} \cdot f_{x_i} = \frac{1}{2}(g_{ii})_{x_j}, & \text{if } \ i \ne j, \\ f_{x_i x_i} \cdot f_{x_j} = (f_{x_i} \cdot f_{x_j})_{x_i} - f_{x_i} \cdot f_{x_j x_i} = (g_{ij})_{x_i} - \frac{1}{2}(g_{ii})_{x_j}, & \text{if } i \ne j \end{cases}$$

$$\begin{cases} f_{x_i x_j} \cdot N = \ell_{ij}, \\ N_{x_i} \cdot f_{x_j} = -\ell_{ij}, \\ N_{x_i} \cdot N = 0. \end{cases} \tag{5.1.3}$$

Let

$$G = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By Proposition 5.1.1, we have

(5.1.4)

$$\begin{cases} P = \begin{pmatrix} g^{11} & g^{12} & 0 \\ g^{12} & g^{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}(g_{11})_{x_1} & \frac{1}{2}(g_{11})_{x_2} & -\ell_{11} \\ (g_{12})_{x_1} - \frac{1}{2}(g_{11})_{x_2} & \frac{1}{2}(g_{22})_{x_1} & -\ell_{12} \\ \ell_{11} & \ell_{12} & 0 \end{pmatrix} \\ Q = \begin{pmatrix} g^{11} & g^{12} & 0 \\ g^{12} & g^{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}(g_{11})_{x_2} & (g_{12})_{x_2} - \frac{1}{2}(g_{22})_{x_1} & -\ell_{12} \\ \frac{1}{2}(g_{22})_{x_1} & \frac{1}{2}(g_{22})_{x_2} & -\ell_{22} \\ \ell_{12} & \ell_{22} & 0 \end{pmatrix} \end{cases}$$

This proves the Proposition. □

Formula (5.1.4) gives explicit formulas for entries of P and Q in terms of g_{ij} and ℓ_{ij}. Moreover, they are related to the Christofell symbols Γ^i_{jk} arise in the geodesic equation (4.2.6) in Theorem 4.2.1. Recall that

$$\Gamma^k_{ij} = \frac{1}{2} g^{km}[ij, m],$$

where (g^{ij}) is the inverse matrix of (g_{ij}), $[ij, k] = g_{ki,j} + g_{jk,i} - g_{ij,k}$, and $g_{ij,k} = \frac{\partial g_{ij}}{\partial x_k}$.

Theorem 5.1.3. *For $1 \leq i, j \leq 2$, we have*

$$p_{ji} = \Gamma^j_{i1}, \qquad q_{ji} = \Gamma^j_{i2} \tag{5.1.5}$$

Proof. Note that (5.1.4) implies

$$\begin{aligned}
p_{11} &= \frac{1}{2}\, g^{11} g_{11,1} + g^{12}(g_{12,1} - \frac{1}{2} g_{11,2}) = \Gamma^1_{11},\\
p_{12} &= \frac{1}{2} g^{11} g_{11,2} + \frac{1}{2} g^{12} g_{22,1} = \Gamma^1_{21},\\
p_{21} &= \frac{1}{2} g^{12} g_{11,1} + g^{22}(g_{12,1} - \frac{1}{2} g_{11,2}) = \Gamma^2_{11},\\
p_{22} &= \frac{1}{2} g^{12} g_{11,2} + \frac{1}{2} g^{22} g_{22,1} = \Gamma^2_{21},\\
q_{11} &= \frac{1}{2} g^{11} g_{11,2} + \frac{1}{2} g^{12} g_{22,1} = \Gamma^1_{12},\\
q_{12} &= g^{11}(g_{12,2} - \frac{1}{2} g_{22,1}) + \frac{1}{2} g^{12} g_{22,2} = \Gamma^1_{22},\\
q_{21} &= \frac{1}{2} g^{12} g_{11,2} + \frac{1}{2} g^{22} g_{22,1} = \Gamma^2_{12},\\
q_{22} &= g^{12}(g_{12,2} - \frac{1}{2} g_{22,1}) + \frac{1}{2} g^{22} g_{22,2} = \Gamma^2_{22}, \qquad (5.1.6)
\end{aligned}$$

□

Note that

$$q_{11} = p_{12}, \quad q_{21} = p_{22}.$$

Theorem 5.1.4. The Fundamental Theorem of surfaces in $\mathbb{R}^3$. *Suppose M is an embedded surface in $\mathbb{R}^3$, and $f : \mathcal{O} \to U$ a local coordinate system on M, and g_{ij}, ℓ_{ij} are the coefficients of* I, II. *Let P, Q be the smooth $gl(3)$-valued maps defined in terms of g_{ij} and ℓ_{ij} by* (5.1.4). *Then P, Q satisfy*

$$P_{x_2} - Q_{x_1} = [P, Q]. \tag{5.1.7}$$

Conversely, let $\mathcal{O}$ be an open subset of $\mathbb{R}^2$, $(g_{ij}), (\ell_{ij}) : \mathcal{O} \to gl(2)$ smooth maps such that (g_{ij}) is positive definite and (ℓ_{ij}) is symmetric, and $P, Q : U \to gl(3)$ the maps defined by (5.1.4). Suppose P, Q satisfies the compatibility equation (5.1.7). Let $(x_1^0, x_2^0) \in \mathcal{O}$, $p_0 \in \mathbb{R}^3$, and u_1, u_2, u_3 a basis of $\mathbb{R}^3$ so that $u_i \cdot u_j = g_{ij}(x_1^0, x_2^0)$ and $u_i \cdot u_3 = 0$ for $1 \leq i, j \leq 2$. Then there exists an open subset $\mathcal{O}_0 \subset \mathcal{O}$ of (x_1^0, x_2^0) and a unique immersion $f : \mathcal{O}_0 \to \mathbb{R}^3$ so that f maps $\mathcal{O}_0$ homeomorphically to $f(\mathcal{O}_0)$ such that

(1) *the first and second fundamental forms of the embedded surface $f(\mathcal{O}_0)$ are given by (g_{ij}) and (ℓ_{ij}) respectively,*
(2) *$f(x_1^0, x_2^0) = p_0$, and $f_{x_i}(x_1^0, x_2^0) = u_i$ for $i = 1, 2$.*

Proof. We have proved the first half of the theorem, and it remains to prove the second half. We assume that P, Q satisfy the compatibility condition (5.1.7). So Frobenius Theorem 2.4.1 implies that the following system has a unique local solution

$$\begin{cases} (v_1, v_2, v_3)_{x_1} = (v_1, v_2, v_3)P, \\ (v_1, v_2, v_3)_{x_2} = (v_1, v_2, v_3)Q, \\ (v_1, v_2, v_3)(x_1^0, x_2^0) = (u_1, u_2, u_3). \end{cases}$$

Since (u_1, u_2, u_3) is invertible, so is (v_1, v_2, v_3).

We claim that $v_i \cdot v_3 = 0$ for $i = 1, 2$. To see this, we let p_{ij} and q_{ij} denote the ij-th entry of P and Q respectively. Then

$$(v_i \cdot v_3)_{x_1} = (v_i)_{x_1} \cdot v_3 + v_i \cdot (v_3)_{x_1} = p_{3i} + p_{i3} = 0.$$

Similar argument implies that $(v_i \cdot v_3)_{x_2} = q_{3i} + q_{i3} = 0$. But $v_i \cdot v_3$ is zero at (x_1^0, x_2^0). So by uniqueness of solution of ODE, we see that $v_i \cdot v_3 = 0$.

Next we want to solve

$$\begin{cases} f_{x_1} = v_1, \\ f_{x_2} = v_2, \\ f(x_1^0, x_2^0) = p_0. \end{cases}$$

The compatibility condition is $(v_1)_{x_2} = (v_2)_{x_1}$. But

$$(v_1)_{x_2} = \sum_{j=1}^3 q_{j1} v_j, \quad (v_2)_{x_1} = \sum_{j=1}^3 p_{j2} v_2.$$

It follows from (5.1.4) that the second column of P is equal to the first column of Q. So $(v_1)_{x_2} = (v_2)_{x_1}$, and hence there exists a unique f. The rest of the theorem follows. □

System (5.1.7) with P, Q defined by (5.1.4) is called the *Gauss-Codazzi equation for the surface* $f(\mathcal{O})$, which is a second order PDE with 9 equations for six functions g_{ij} and ℓ_{ij}. Equaqtion (5.1.7) is too complicated to memorize. It is more useful and simpler to just remember how to derive the Gauss-Codazzi equation.

It follows from (5.1.1), (5.1.4), and (5.1.5) that we have

$$f_{x_i x_1} = \sum_{j=1}^{2} p_{ji} f_{x_j} + \ell_{i1} N = \sum_{j=1}^{2} \Gamma^j_{i1} f_{x_j} + \ell_{i1} N,$$
$$f_{x_i x_2} = \sum_{j=1}^{2} q_{j2=i} f_{x_j} + \ell_{i2} N = \sum_{j=1}^{2} \Gamma^j_{i2} f_{x_j} + \ell_{i2} N,$$

where p_{ij} and q_{ij} are defined in (5.1.4).

So we have

$$f_{x_i x_j} = \Gamma^1_{ij} f_{x_1} + \Gamma^2_{ij} f_{x_2} + \ell_{ij} N, \tag{5.1.8}$$

Proposition 5.1.5. *Let $f : \mathcal{O} \to \mathbb{R}^3$ be a local coordinate system of an embedded surface M in $\mathbb{R}^3$, and $\alpha(t) = f(x_1(t), x_2(t))$. Then α satisfies the geodesic equation* (4.2.6) *if and only if $\alpha''(t)$ is normal to M at $\alpha(t)$ for all t.*

Proof. Differentiate α' to get $\alpha' = \sum_{i=1}^{2} f_{x_i} x_i'$. So

$$\begin{aligned} \alpha'' &= \sum_{i,j=1}^{2} f_{x_i x_j} x_i' x_j' + f_{x_i} x_i'' \\ &= \sum_{i,j,k=1}^{2} \Gamma^k_{ij} f_{x_k} x_i' x_j' + \ell_{ij} N + f_{x_i} x_i'' \\ &= \sum_{i,j=1}^{2} (\Gamma^k_{ij} x_i' x_j' + x_k'') f_{x_k} + \ell_{ij} N = 0 + \ell_{ij} N = \ell_{ij} N. \end{aligned}$$

□

5.2. **The Gauss Theorem.**

Equation (5.1.7) is the *Gauss-Codazzi equation* for M.

The Gaussian curvature K is defined to be the determinant of the shape operator $-dN$, which depends on both the first and second fundamental forms of the surface. In fact, by Proposition 3.4.2

$$K = \frac{\ell_{11}\ell_{22} - \ell_{12}^2}{g_{11}g_{22} - g_{12}^2}.$$

We will show below that K can be computed in terms of g_{ij} alone. Equate the 12 entry of equation (5.1.7) to get

$$(p_{12})_{x_2} - (q_{12})_{x_1} = \sum_{j=1}^{3} p_{1j}q_{j2} - q_{1j}p_{j2}.$$

Recall that formula (5.1.6) gives $\{p_{ij}, q_{ij} \mid 1 \leq i, j \leq 2\}$ in terms of the first fundamental form I. We move terms involves p_{ij}, q_{ij} with $1 \leq i, j \leq 2$ to one side to get

$$(5.2.1) \qquad (p_{12})_{x_2} - (q_{12})_{x_1} - \sum_{j=1}^{2} p_{1j}q_{j2} - q_{1j}p_{j2} = p_{13}q_{32} - q_{13}p_{32}.$$

We claim that the right hand side of (5.2.1) is equal to

$$-g^{11}(\ell_{11}\ell_{22} - \ell_{12}^2) = -g^{11}(g_{11}g_{22} - g_{12}^2)K.$$

To prove this claim, use (5.1.4) to compute P, Q to get

$$p_{13} = -(g^{11}\ell_{11} + g^{12}\ell_{12}), \quad p_{32} = \ell_{12},$$
$$q_{13} = -(g^{11}\ell_{12} + g^{12}\ell_{22}), \quad q_{32} = \ell_{22}.$$

So we get

$$(5.2.2) \quad (p_{12})_{x_2} - (q_{12})_{x_1} - \sum_{j=1}^{2} p_{1j}q_{j2} - q_{1j}p_{j3} = g^{11}(g_{11}g_{22} - g_{12}^2)K.$$

Hence we have proved the claim and also obtained a formula of K purely in terms of g_{ij} and their derivatives:

$$K = \frac{(p_{12})_{x_2} - (q_{12})_{x_1} - \sum_{j=1}^{2} p_{1j}q_{j2} - q_{1j}p_{j3}}{g^{11}(g_{11}g_{22} - g_{12}^2)}.$$

This proves

Theorem 5.2.1. Gauss Theorem. *The Gaussian curvature of a surface in $\mathbb{R}^3$ can be computed from the first fundamental form.*

The equation (5.2.2), obtained by equating the 12-entry of (5.1.7), is the *Gauss equation*.

A geometric quantity on an embedded surface M in $\mathbb{R}^n$ is called *intrinsic* if it only depends on the first fundamental form I. Otherwise, the property is called *extrinsic*, i.e., it depends on both I and II.

We have seen that the Gaussian curvature and geodesics are intrinsic quantities, and the mean curvature is extrinsic.

If $\phi : M_1 \to M_2$ is a diffeomorphism and $f(x_1, x_2)$ is a local coordinates on M_1, then $\phi \circ f(x_1, x_2)$ is a local coordinate system of M_2. The diffeomorphism ϕ is an *isometry* if the first fundamental forms for M_1, M_2 are the same written in terms of dx_1, dx_2. In particular,

(i) ϕ preserves anlges and arc length, i.e., the arc length of the curve $\phi(\alpha)$ is the same as the curve α and the angle between the curves $\phi(\alpha)$ and $\phi(\beta)$ is the same as the angle between α and β,

(ii) ϕ maps geodesics to geodesics.

Euclidean plane geometry studies the geometry of triangles. Note that triangles can be viewed as a triangle in the plane with each side being a geodesic. So a natural definition of a triangle on an embedded surface M is a piecewise smooth curve with three geodesic sides and any two sides meet at an angle lie in $(0, \pi)$. One important problem in geometric theory of M is to understand the geometry of triangles on M. For example, what is the sum of interior angles of a triangle on an embedded surface M? This will be answered by the Gauss-Bonnet Theorem.

Note that the first fundamental forms for the plane

$$f(x_1, x_2) = (x_1, x_2, 0)$$

and the cylinder

$$h(x_1, x_2) = (\cos x_1, \sin x_1, x_2)$$

have the same and is equal to $\mathrm{I} = dx_1^2 + dx_2^2$, and both surfaces have constant zero Gaussian curvature (cf. Examples 3.4.3 and 3.4.5). We have also proved that geodesics are determined by I alone. So the geometry of triangles on the cylinder is the same as the geometry of triangles in the plane. For example, the sum of interior anlges of a triangle on the plane (and hence on the cylinder) must be π. In fact, let ϕ denote the map from $(0, 2\pi) \times \mathbb{R}$ to the cylinder minus the line $(1, 0, x_2)$ defined by

$$\phi(x_1, x_2, 0) = (\cos x_1, \sin x_1, x_2).$$

Then ϕ is an isometry.

5.3. **Gauss-Codazzi equation in orthogonal coordinates.**

If the local coordinates x_1, x_2 are orthogonal, i.e., $g_{12} = 0$, then the Gauss-Codazzi equation (5.1.7) becomes much simplier. Instead of putting $g_{12} = 0$ to (5.1.7), we derive the Gauss-Codazzi equation directly using an o.n. moving frame. We write

$$g_{11} = A_1^2, \quad g_{22} = A_2^2, \quad g_{12} = 0.$$

Let

$$e_1 = \frac{f_{x_1}}{A_1}, \quad e_2 = \frac{f_{x_2}}{A_2}, \quad e_3 = N.$$

Then (e_1, e_2, e_3) is an o.n. moving frame on M. Write

$$\begin{cases} (e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3)\tilde{P}, \\ (e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3)\tilde{Q}. \end{cases}$$

Since (e_1, e_2, e_3) is orthogonal, $\tilde{P}, \tilde{Q}$ are skew-symmetric. Moreover,

$$\tilde{p}_{ij} = (e_j)_{x_1} \cdot e_i, \quad \tilde{q}_{ij} = (e_j)_{x_2} \cdot e_i.$$

A direct computation gives

$$\begin{aligned} (e_1)_{x_1} \cdot e_2 &= \left(\frac{f_{x_1}}{A_1}\right)_{x_1} \cdot \frac{f_{x_2}}{A_2} = \frac{f_{x_1x_1} \cdot f_{x_2}}{A_1A_2} \\ &= \frac{(f_{x_1} \cdot f_{x_2})_{x_1} - f_{x_1} \cdot f_{x_1x_2}}{A_1A_2} \\ &= -\frac{(\frac{1}{2}A_1^2)_{x_2}}{A_1A_2} = -\frac{(A_1)_{x_2}}{A_2}. \end{aligned}$$

Similar computation gives the coefficients $\tilde{p}_{ij}$ and $\tilde{q}_{ij}$:
(5.3.1)

$$\tilde{P} = \begin{pmatrix} 0 & \frac{(A_1)_{x_2}}{A_2} & -\frac{\ell_{11}}{A_1} \\ -\frac{(A_1)_{x_2}}{A_2} & 0 & -\frac{\ell_{12}}{A_2} \\ \frac{\ell_{11}}{A_1} & \frac{\ell_{12}}{A_2} & 0 \end{pmatrix}, \qquad \tilde{Q} = \begin{pmatrix} 0 & -\frac{(A_2)_{x_1}}{A_1} & -\frac{\ell_{12}}{A_1} \\ \frac{(A_2)_{x_1}}{A_1} & 0 & -\frac{\ell_{22}}{A_2} \\ \frac{\ell_{12}}{A_1} & \frac{\ell_{22}}{A_2} & 0 \end{pmatrix}$$

To get the Gauss-Codazzi equation of the surface parametrized by an orthogonal coordinates we only need to compute the 21-th, 31-th, and 32-the entry of the following equation

$$(\tilde{P})_{x_2} - (\tilde{Q})_{x_1} = [\tilde{P}, \tilde{Q}],$$

and we obtain

$$(5.3.2)\qquad \begin{cases} -\left(\frac{(A_1)_{x_2}}{A_2}\right)_{x_2} - \left(\frac{(A_2)_{x_1}}{A_1}\right)_{x_1} = \frac{\ell_{11}\ell_{22}-\ell_{12}^2}{A_1A_2}, \\ \left(\frac{\ell_{11}}{A_1}\right)_{x_2} - \left(\frac{\ell_{12}}{A_1}\right)_{x_1} = \frac{\ell_{12}(A_2)_{x_1}}{A_1A_2} + \frac{\ell_{22}(A_1)_{x_2}}{A_2^2} \\ \left(\frac{\ell_{12}}{A_2}\right)_{x_2} - \left(\frac{\ell_{22}}{A_2}\right)_{x_1} = -\frac{\ell_{11}(A_2)_{x_1}}{A_1^2} - \frac{\ell_{12}(A_1)_{x_2}}{A_1A_2}. \end{cases}$$

The first equation of (5.3.2) is called the *Gauss equation*. Note that the Gaussian curvature is

$$K = \frac{\ell_{11}\ell_{22}-\ell_{12}^2}{(A_1A_2)^2}.$$

So we have

$$(5.3.3)\qquad K = -\frac{\left(\frac{(A_1)_{x_2}}{A_2}\right)_{x_2} + \left(\frac{(A_2)_{x_1}}{A_1}\right)_{x_1}}{A_1A_2}.$$

We have seen that the Gauss-Codazzi equation becomes much simpler in orthogonal coordinates. Can we always find local orthogonal coordinates on a surface in $\mathbb{R}^3$? This question can be answered by the following theorem, which we state without a proof.

Theorem 5.3.1. *Suppose $f : \mathcal{O} \to \mathbb{R}^3$ be a surface, $x_0 \in \mathcal{O}$, and $Y_1, Y_2 : \mathcal{O} \to \mathbb{R}^3$ smooth maps so that $Y_1(x_0), Y_2(x_0)$ are linearly independent and tangent to $M = f(\mathcal{O})$ at $f(x_0)$. Then there exist open subset $\mathcal{O}_0$ of $\mathcal{O}$ containing x_0, open subset $\mathcal{O}_1$ of $\mathbb{R}^2$, and a diffeomorphism $h : \mathcal{O}_1 \to \mathcal{O}_0$ so that $(f \circ h)_{y_1}$ and $(f \circ h)_{y_2}$ are parallel to $Y_1 \circ h$ and $Y_2 \circ h$.*

The above theorem says that if we have two linearly independent vector fields Y_1, Y_2 on a surface, then we can find a local coordinate system $\phi(y_1, y_2)$ so that ϕ_{y_1}, ϕ_{y_2} are parallel to Y_1, Y_2 respectively.

Given an arbitrary local coordinate system $f(x_1, x_2)$ on M, we apply the Gram-Schmidt process to f_{x_1}, f_{x_2} to construct smooth o.n. vector fields e_1, e_2:

$$e_1 = \frac{f_{x_1}}{\sqrt{g_{11}}},$$
$$e_2 = \frac{\sqrt{g_{11}}(f_{x_2} - \frac{g_{12}}{g_{11}} f_{x_1})}{\sqrt{g_{11}g_{22} - g_{12}^2}},$$

By Theorem 5.3.1, there exists new local coordinate system $\tilde{f}(y_1, y_2)$ so that $\frac{\partial \tilde{f}}{\partial x_1}$ and $\frac{\partial \tilde{f}}{\partial x_2}$ are parallel to e_1 and e_2. So the first fundamental form written in this coordinate system has the form

$$\tilde{g}_{11} dy_1^2 + \tilde{g}_{22} dy_2^2.$$

However, in general we can not find coordinate system $\hat{f}(y_1, y_2)$ so that e_1 and e_2 are coordinate vector fields $\frac{\partial \hat{f}}{\partial y_1}$ and $\frac{\partial \hat{f}}{\partial y_2}$ because if we can then the first fundamental form of the surface is $\mathrm{I} = dy_1^2 + dy_2^2$, which implies that the Gaussian curvature of the surface must be zero.

5.4. **Line of curvature coordinates.**

Suppose M is an embedded surface in $\mathbb{R}^3$, $f : \mathcal{O} \to U \subset M$ a local coordinate system, and $p_0 = f(x^0)$ is not an umbilic point. This means that the shape operator dN_{p_0} has two distinct eigenvalues. Since dN is smooth, there exists an open subset $\mathcal{O}_0$ of $\mathcal{O}$ containing x^0 so that $f(x)$ is not umbilic for all $x \in \mathcal{O}_0$. We can use linear algebra to write down a formula for two eigenvectors v_1, v_2 of dN on $\mathcal{O}_0$ and see that v_1, v_2 are smooth on $\mathcal{O}_0$. Since dN_p is self-adjoint with distinct eigenvalues, v_1 and v_2 are perpendicular. By Theorem 5.3.1, we can change coordinates so that the coordinate vector fields are parallel to v_1 and v_2. This means that we may change coordinates $\tilde{f} : \mathcal{O}_1 \to U_1 \subset M \subset \mathbb{R}^3$ so that $\frac{\partial \tilde{f}}{\partial y_1}$ and $\frac{\partial \tilde{f}}{\partial y_2}$ are parallel to v_1, v_2 respectively. In this new coordinates,

$$\tilde{g}_{12} = \tilde{f}_{y_1} \cdot \tilde{f}_{y_2} = 0$$

because $v_1 \cdot v_2 = 0$. But since $(\tilde{f})_{y_1}$ and $(\tilde{f})_{y_2}$ are in the principal directions, the shape operator is diagonalized with respect to this eigenbasis. Hence

$$\tilde{\ell}_{12} = 0.$$

In other words, both I, II are diagonalized. We call such coordinates *line of curvature coordinates*.

Therefore we have proved

Theorem 5.4.1. *If $M \subset \mathbb{R}^3$ is an embedded surface and $p_0 \in M$ is not umbilic, then there exists a local line of curvature coordinate system $f : \mathcal{O} \to U \subset M \subset$ near p_0, i.e.,*

$$g_{12} = f_{x_1} \cdot f_{x_2} = 0, \quad \ell_{12} = f_{x_1 x_2} \cdot N = 0,$$

or equivalently,

$$\mathrm{I} = A_1^2 \; dx_1^2 + A_2^2 \; dx_2^2, \quad \mathrm{II} = \ell_{11} \; dx_1^2 + \ell_{22} \; dx_2^2.$$

Theorem 5.4.2. *Suppose $f(x_1, x_2)$ is a local line of curvature coordinate system on an embedded surface M in $\mathbb{R}^3$ with* I, II *as in Theorem*

5.4.1. Then the Gauss-Codazzi equation is

$$\left(\frac{(A_1)_{x_2}}{A_2}\right)_{x_2} + \left(\frac{(A_2)_{x_1}}{A_1}\right)_{x_1} = -\frac{\ell_{11}\ell_{22}}{A_1A_2}, \tag{5.4.1}$$

$$\left(\frac{\ell_{11}}{A_1}\right)_{x_2} = \frac{\ell_{22}(A_1)_{x_2}}{A_2^2} \tag{5.4.2}$$

$$\left(\frac{\ell_{22}}{A_2}\right)_{x_1} = \frac{\ell_{11}(A_2)_{x_1}}{A_1^2}. \tag{5.4.3}$$

Proof. This theorem can be proved either by substituting $\ell_{12} = 0$ into (5.3.2) or derive directly. But we will derive it directly. Let e_1, e_2, e_3 be the o.n. basis so that

$$e_1 = \frac{f_{x_1}}{A_1}, \quad e_2 = \frac{f_{x_2}}{A_2}, \quad e_3 = N,$$

and $(e_i)_{x_1} \cdot e_j = p_{ji}$, $(e_i)_{x_2} \cdot e_j = q_{ij}$. A direct computation gives

$$\begin{cases} (e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3) \begin{pmatrix} 0 & \frac{(A_1)_{x_2}}{A_2} & -\frac{\ell_{11}}{A_1} \\ -\frac{(A_1)_{x_2}}{A_2} & 0 & 0 \\ \frac{\ell_{11}}{A_1} & 0 & 0 \end{pmatrix}, \\ (e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3) \begin{pmatrix} 0 & -\frac{(A_2)_{x_1}}{A_1} & 0 \\ \frac{(A_2)_{x_1}}{A_1} & 0 & -\frac{\ell_{22}}{A_2} \\ 0 & \frac{\ell_{22}}{A_2} & 0 \end{pmatrix}. \end{cases} \tag{5.4.4}$$

The Gauss-Codazzi equation is $P_{x_2} - Q_{x_1} = [P, Q]$. Since P, Q are skew symmetric, we get 3 equations (the 12, 13, and 23 entries) as stated in the theorem. □

6. Surfaces in $\mathbb{R}^3$ with $K = -1$

We will show that there exists line of curvature coordinates $f(x, y)$ so that the angle u between asymptotic lines satisfies the sine-Gordon equation (SGE):

$$u_{xx} - u_{yy} = \sin u.$$

In fact, we prove that there is a one to one correspondence between local solutions u of the SGE with $\text{Im}(u) \subset (0, \pi)$ and local surfaces in $\mathbb{R}^3$ with $K = -1$ up to rigid motions. We also prove

(1) the classical Bäcklund transformations that generate infinitely many surfaces of $K=-1$ in $\mathbb{R}^3$ from a given one by solving a system of compatible ODEs.
(2) the Hilbert Theorem that there is no complete $K=-1$ surfaces in $\mathbb{R}^3$.

6.1. $K=-1$ surfaces in $\mathbb{R}^3$ and the sine-Gordon equation.

Suppose M be a surface in $\mathbb{R}^3$ with $K=-1$, and λ_1, λ_2 are the two principal curvatures. Since $K=\lambda_1\lambda_2=-1$, $\lambda_1 \neq \lambda_2$, i.e., there are no umbilic points. Hence we can find local line of curvature coordinates $f(x_1, x_2)$ on M. Assume the fundamental forms are

$$\mathrm{I} = A_1^2\ dx_1^2 + A_2^2\ dx_2^2, \quad \mathrm{II} = \ell_{11}\ dx_1^2 + \ell_{22}\ dx_2^2.$$

Then the prinicpal curvatures are

$$\lambda_1 = \frac{\ell_{11}}{A_1^2}, \quad \lambda_2 = \frac{\ell_{22}}{A_2^2}.$$

But $\lambda_1\lambda_2 = -1$, so we may assume that there exists a smooth function q so that

$$\lambda_1 = \frac{\ell_{11}}{A_1^2} = \tan q, \qquad \lambda_2 = \frac{\ell_{22}}{A_2^2} = -\cot q,$$

i.e.,

$$\frac{\ell_{11}}{A_1} = A_1 \tan q, \quad \frac{\ell_{22}}{A_2} = -A_2 \cot q. \tag{6.1.1}$$

Since $A_1, A_2, \ell_{11}, \ell_{22}$ satisfy the Gauss-Codazzi equation, substitue (6.1.1) to the Codazzi equations (5.4.2) and (5.4.3) to get

$$(A_1 \tan q)_{x_2} = -\cot q\ (A_1)_{x_2},$$
$$(-A_2 \cot q)_{x_1} = \tan q\ (A_2)_{x_1}.$$

Compute directly to get

$$(A_1)_{x_2} \tan q + A_1 \sec^2 q\ q_{x_2} = -\cot q\ (A_1)_{x_2},$$

which implies that

$$(\tan q + \cot q)(A_1)_{x_2} = -A_1(\sec^2 q)\ q_{x_2}.$$

So we obtain

$$\frac{(A_1)_{x_2}}{A_1} = -\frac{\sin q}{\cos q}\ q_{x_2}.$$

Use a similar computation to get

$$\frac{(A_2)_{x_1}}{A_2} = \frac{\cos q}{\sin q}\ q_{x_1}.$$

In other words, we have

$$(\log A_1)_{x_2} = (\log\cos q)_{x_2}, \quad (\log A_2)_{x_1} = (\log\sin q)_{x_1}.$$

Hence there exist $c_1(x_1)$ and $c_2(x_2)$ so that

$$\log A_1 = \log\cos q + c_1(x_1), \quad \log A_2 = \log\sin q + c_2(x_2),$$

i.e.,

$$A_1 = e^{c_1(x_1)}\cos q, \quad A_2 = e^{c_2(x_2)}\sin q.$$

Because I is positive definite, A_1, A_2 never vanishes. So we may assume both $\sin q$ and $\cos q$ are positive, i.e., $q \in (0, \frac{\pi}{2})$. Now change coordinates to $(\tilde{x}_1(x_1), \tilde{x}_2(x_2))$ so that

$$\frac{d\tilde{x}_1}{dx_1} = e^{c_1(x_1)}, \quad \frac{d\tilde{x}_2}{dx_2} = e^{c_2(x_2)}.$$

Since

$$\frac{\partial f}{\partial \tilde{x}_1} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial \tilde{x}_1} = \frac{\partial f}{\partial x_1}e^{-c_1(x_1)},$$

$||f_{\tilde{x}_1}|| = \cos q$. Similar calculation implies that $||f_{\tilde{x}_2}|| = \sin q$. Since $\tilde{x}_i$ is a function of $\tilde{x}_i$ alone, $f_{\tilde{x}_i}$ is parallel to f_{x_i}. So $(\tilde{x}_1, \tilde{x}_2)$ is also a line of curvature coordinate system and the coefficients of II in $(\tilde{x}_1, \tilde{x}_2)$ coordinate system is

$$\tilde{\ell}_{11} = \tan q\cos^2 q = \sin q\cos q, \quad \tilde{\ell}_{22} = -\cot q\sin^2 q = -\sin q\cos q.$$

Therefore, we have proved part of the following Theorem:

Theorem 6.1.1. *Let M^2 be a surface in $\mathbb{R}^3$ with $K = -1$. Then locally there exists line of curvature coordinates x_1, x_2 so that*

$$\mathrm{I} = \cos^2 q\ dx_1^2 + \sin^2 q\ dx_2^2, \quad \mathrm{II} = \sin q\cos q\ (dx_1^2 - dx_2^2), \tag{6.1.2}$$

where $2q$ is the angle between two asymptotic directions. Moreover, the Gauss-Codazzi equation is the sine-Gordon equation

$$q_{x_1x_1} - q_{x_2x_2} = \sin q\cos q. \tag{6.1.3}$$

Proof. It remains to compute the Gauss-Codazzi equation. Let

$$e_1 = \frac{f_{x_1}}{\cos q}, \quad e_2 = \frac{f_{x_2}}{\sin q}, \quad e_3 = N = \frac{f_{x_1}\times f_{x_2}}{||f_{x_1}\times f_{x_2}||}.$$

Write

$$(e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3)P, \quad (e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3)Q.$$

We have given formulas for P, Q in (5.4.4) when (x_1, x_2) is a line of curvature coordinate system. Since $A_1 = \cos q$, $A_2 = \sin q$, and $\ell_{11} = -\ell_{22} = \sin q \cos q$, we have

(6.1.4)
$$P = \begin{pmatrix} 0 & -q_{x_2} & -\sin q \\ q_{x_2} & 0 & 0 \\ \sin q & 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & -q_{x_1} & 0 \\ q_{x_1} & 0 & \cos q \\ 0 & -\cos q & 0 \end{pmatrix}.$$

A direct computation shows that Codazzi equations are satified automatically, i.e., the 13 and 23 entries of $P_{x_2} - Q_{x_1}$ are equal to the 13 and 23 entries of $[P, Q]$ respectively. The Gauss equation (equating the 12 entry) gives

$$-q_{x_2x_2} + q_{x_1x_1} = \sin q \cos q.$$

Since $\mathrm{II} = \sin q \cos q\ (dx_1^2 - dx_2^2)$, $f_{x_1} \pm f_{x_2}$ are asymptotic directions. Use I to see that $f_{x_1} \pm f_{x_2}$ are unit vectors. Since

$$(f_{x_1} + f_{x_2}) \cdot (f_{x_1} - f_{x_2}) = \cos^2 q - \sin^2 q = \cos(2q),$$

the angle between the asymptotic directions $f_{x_1} + f_{x_2}$ and $f_{x_1} - f_{x_2}$ is $2q$. □

A consequence of the proof of the above theorem is

Corollary 6.1.2. *Let P, Q be as in* (6.1.4). *Then system*

$$(e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3)P, \quad (e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3)Q$$

is solvable if and only if q satisfies the SGE (6.1.3).

Let $O(3)$ denote the space of all orthogonal 3×3 matrices, and $o(3)$ the space of all skew-symmetric 3×3 matrices. A map $g : \mathbb{R}^2 \to O(3)$ is smooth if $i \circ g : \mathbb{R}^2 \to gl(3)$ is smooth, where $i : O(3) \to gl(3)$ is the inclusion.

It follows from the Fundamental Theorem of surfaces and Corollary 6.1.2 that The converse of Theorem 6.1.1 is true :

Theorem 6.1.3. *Let $q : \mathcal{O} \to \mathbb{R}$ be a solution of the SGE* (6.1.3), *$p_0 \in \mathbb{R}^3$, $(x_1^0, x_2^0) \in \mathcal{O}$, and u_1, u_2, u_3 an o.n. basis. Let $P, Q : \mathcal{O} \to o(3)$ be the maps defined by* (6.1.4). *Then there exists an open subset $\mathcal{O}_1$ of (x_1^0, x_2^0) in $\mathcal{O}$ and a unique solution $(f, e_1, e_2, e_3) : \mathcal{O}_1 \to \mathbb{R}^3 \times O(3)$ for the following system*

$$\text{(6.1.5)} \qquad \begin{cases} (e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3)P, \\ (e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3)Q, \\ f_{x_1} = \cos q\ e_1, \\ f_{x_2} = \sin q\ e_2, \\ f(x_1^0, x_2^0) = p_0, \quad e_1(x_1^0, x_2^0) = u_1, \quad e_2(x_1^0, x_2^0) = u_2. \end{cases}$$

Moreover, if $\sin q \cos q > 0$ *on* $\mathcal{O}_1$*, then* $f(\mathcal{O}_1)$ *is an immersed surface with* $K = -1$ *and the two fundamental forms are of the form* (6.1.2).

In other words, there is a 1-1 correspondence between solutions q of the SGE (6.1.3) with $\text{Im}(q) \subset (0, \frac{\pi}{2})$ and local surfaces of $\mathbb{R}^3$ with $K = -1$ up to rigid motion.

Let $f(x_1, x_2)$ be a local line of curvature coordinate system given in Theorem 6.1.1 of an embedded surface M in $\mathbb{R}^3$ with $K = -1$, and q the corresponding solution of the SGE (6.1.3). The SGE is a non-linear wave equation, and (x_1, x_2) is the space-time coordinate. We have proved that $f_{x_1} \pm f_{x_2}$ are asymptotic directions. If we make a change of coordinates $x_1 = s + t$, $x_2 = s - t$, then $f_s = f_{x_1} + f_{x_2}$ and $f_t = f_{x_1} - f_{x_2}$. A direct computation shows that the two fundamental forms written in (s, t) coordinates are

$$\begin{cases} \text{I} = ds^2 + 2\cos 2q \; dsdt + dt^2, \\ \text{II} = \sin 2q \; ds \; dt, \end{cases} \tag{6.1.6}$$

and the SGE (6.1.3) becomes

$$2q_{st} = \sin(2q). \tag{6.1.7}$$

A local coordinate system (x, y) on a surface $f(x, y)$ in $\mathbb{R}^3$ is called an *asymptotic coordinate system* if f_x, f_y are parallel to the asymptotic direction, i.e., $\ell_{11} = \ell_{22} = 0$. Note that the (s, t) coordinate constructed above for $K = -1$ surfaces in $\mathbb{R}^3$ is an asymptotic coordinate system and s, t are arc length parameter. This coordinate system is called the *Tchbyshef coordinate system for* $K = -1$ *surfaces in* $\mathbb{R}^3$.

6.2. **Moving frame method.**

If P, Q are skew-symmetric, then so is $[P, Q] = PQ - QP$ because

$$(PQ - QP)^t = Q^tP^t - P^tQ^t = (-Q)(-P) - (-P)(-Q) = -(PQ - QP).$$

We use Cartan's method of moving frames to derive the local theory of surfaces in $\mathbb{R}^3$. This method uses differential forms instead of vector fields to derive the Gauss-Codazzi equation.

Proposition 6.2.1. *Let* $\mathcal{O}$ *be an open subset of* $\mathbb{R}^2$*, and* $P = (p_{ij}), Q = (q_{ij})$ *smooth maps from* $\mathcal{O}$ *to* $o(3)$*. Let* $w_{ij} = p_{ij}dx_1 + q_{ij}dx_2$*. Then the following statements are equivalent:*

(i) $\begin{cases} g_{x_1} = gP, \\ g_{x_2} = gQ \end{cases}$ *is solvable,*

(ii) $-P_{x_2} + Q_{x_1} = [P, Q]$,

(iii)

$$dw_{ij} = -\sum_{k=1}^{3} w_{ik} \wedge w_{kj} \tag{6.2.1}$$

Proof. The equivalence of (i) and (ii) is given by Corollary 2.4.2. It remains to prove (ii) if and only if (iii). A direct computation gives

$$\begin{aligned} dw_{ij} &= d(p_{ij}dx_1 + q_{ij}dx_2) = (-(p_{ij})_{x_2} + (q_{ij})_{x_1})\, dx_1 \wedge dx_2, \quad \text{and} \\ -\sum_k w_{ik} \wedge w_{kj} &= -\sum_k (p_{ik}dx_1 + q_{ik}dx_2) \wedge (p_{kj}dx_1 + q_{kj}dx_2) \\ &= -\sum_k (p_{ik}q_{kj} - q_{ik}p_{kj}) dx_1 \wedge dx_2 \\ &= -[P, Q]_{ij} dx_1 \wedge dx_2. \end{aligned}$$

So (ii) and (iii) are equivalent. □

We introduce some notations. Let w_{ij} be 1-forms for $1 \le i, j \le 3$. Then $w = (w_{ij})$ is called a *$gl(3)$-valued 1-form.* Let $\tau = (\tau_{ij})$ be a $gl(3)$-valued 1-form. Let $\tau \wedge w$ denote the $gl(3)$-valued 1-form whose ij-th entry is $\sum_{k=1}^{3} \tau_{ik} \wedge w_{kj}$. Use matrix notation, (6.2.1) can be written as

$$dw = -w \wedge w, \tag{6.2.2}$$

which is called the *Maurer-Cartan equation.*

Once we get used to computations involving matrix valued maps and differential forms, we can obtain the Maurer-Cartan equation easily as follows: Because $w = g^{-1}dg$, we have

$$dw = d(g^{-1}) \wedge dg + g^{-1}d(dg) = -g^{-1}dg\; g^{-1} \wedge dg + 0 = -w \wedge w,$$

i.e., w satisfies (6.2.2).

Since $w_{ij} = -w_{ji}$, $w_{ii} = 0$. So (6.2.1) becomes

$$\begin{cases} dw_{12} = -w_{13} \wedge w_{32}, \\ dw_{13} = -w_{12} \wedge w_{23}, \\ dw_{23} = -w_{21} \wedge w_{13}, \end{cases} \tag{6.2.3}$$

Lemma 6.2.2. Cartan's Lemma *If θ_1, θ_2 are linearly independent 1-forms on an open subset $\mathcal{O}$ of $\mathbb{R}^2$ at each point of $\mathcal{O}$, then there exists a unique θ_{12} satisfies*

$$\begin{cases} d\theta_1 = -\theta_{12} \wedge \theta_2, \\ d\theta_2 = -\theta_{21} \wedge \theta_1, \end{cases} \quad \text{where } \theta_{12} + \theta_{21} = 0. \tag{6.2.4}$$

Proof. Suppose $\theta_i = A_i\, dx + B_i\, dy$ with $i = 1, 2$, and $\theta_{12} = p\, dx + q\, dy$. Then (6.2.4) implies that

$$\begin{cases} -pB_2 + qA_2 = -(A_1)_y + (B_1)_x, \\ pB_1 - qA_1 = -(A_2)_y + (B_2)_x. \end{cases} \tag{6.2.5}$$

Since θ_1, θ_2 are linearly independent at every point, $A_1B_2 - A_2B_1 \neq 0$. So the linear system (6.2.5) has a unique solution p, q. □

Let $f : \mathcal{O} \to U \subset M$ be a local coordinate system of the embedded surface M in $\mathbb{R}^3$. When we view f as a map from $\mathcal{O}$ to $\mathbb{R}^3$, the differential of f is

$$df = f_{x_1} dx_1 + f_{x_2} dx_2.$$

Let $\pi_i : M \to \mathbb{R}^3$ be the function defined by $\pi_i(y_1, y_2, y_3) = y_i$. The differential

$$(d\pi_i)_{f(q)} = d(f_i)_q.$$

Let $\pi = (\pi_1, \pi_2, \pi_3) : M \to \mathbb{R}^3$. Then π is the inclusion map, and

$$d\pi_{f(q)} = df_q = f_{x_1} dx_1 + f_{x_2} dx_2.$$

Since dx_1, dx_2 are 1-forms dual to the basis f_{x_1}, f_{x_2} of TM, the restriction of $d\pi$ to each TM_p is the identity map. Suppose e_1, e_2 are smooth orthonormal vector fields of M defined on $f(\mathcal{O})$, and w_1, w_2 are dual 1-form on M, i.e., $w_i(e_j) = \delta_{ij}$. Since $df_q = d\pi_{f(q)}$ is the identiy map, by Corollary 3.1.7 we have

$$df = \theta_1 e_1 + \theta_2 e_2.$$

Let $e_3 = N$ be the unit normal vector field. Set

$$w_{ij} = de_i \cdot e_j, \quad 1 \leq i, j \leq 3. \tag{6.2.6}$$

Let $g = (e_1, e_2, e_3)$. Rewrite (6.2.6) as

$$dg = gw.$$

Differentiate $e_i \cdot e_j = \delta_{ij}$ to get

$$w_{ij} + w_{ji} = 0,$$

i.e., $w = (w_{ij})$ is a $o(3)$-valued 1-form. The Gauss-Codazzi equation is the compatibility condition for $dg = gw$. So it follows from Proposition

6.2.1 that the Gauss-Codazzi equation for M is the Maurer-Cartan equation $dw = -w \wedge w$. Since $w_{ii} = 0$, the Maurer-Cartan equation is

$$\begin{cases} dw_{12} = -w_{13} \wedge w_{32}, \\ dw_{13} = -w_{12} \wedge w_{23}, \\ dw_{23} = -w_{21} \wedge w_{13}. \end{cases} \tag{6.2.7}$$

Differentiate $df = w_1 e_1 + w_2 e_2$ to get

$$\begin{aligned} 0 = d(df) &= dw_1 e_1 - w_1 \wedge de_1 + dw_2 e_2 - w_2 \wedge de_2 \\ &= dw_1 e_1 - w_1 \wedge (w_{21} e_2 + w_{31} e_3) + dw_2 e_2 - w_2 \wedge (w_{12} e_1 + w_{32} e_3) \\ &= (dw_1 - w_2 \wedge w_{12}) e_1 + (dw_2 - w_1 \wedge w_{21}) e_2 - (w_1 \wedge w_{31} + w_2 \wedge w_{32}) e_3. \end{aligned}$$

So we have
(6.2.8)
$$(dw_1 - w_2 \wedge w_{12}) e_1 + (dw_2 - w_1 \wedge w_{21}) e_2 - (w_1 \wedge w_{31} + w_2 \wedge w_{32}) e_3 = 0$$

The coefficients of e_1, e_2 are zero imply that

$$\begin{cases} dw_1 - w_2 \wedge w_{12} = 0, \\ dw_2 - w_1 \wedge w_{21} = 0, \end{cases} \tag{6.2.9}$$

where $w_{12} = -w_{21}$. Equation (6.2.9) is called the *structure equation*. It follows from the Cartan Lemma 6.2.2 that w_{12} can be computed in terms of w_1, w_2. But the first fundamental form is

$$\mathrm{I} = w_1 \otimes w_1 + w_2 \otimes w_2.$$

So this proves that w_{12} only depends on I.

The coefficient of e_3 in equation (6.2.8) is zero implies that

$$w_1 \wedge w_{31} + w_2 \wedge w_{32} = 0. \tag{6.2.10}$$

Write

$$w_{31} = h_{11} w_1 + h_{12} w_2, \quad w_{32} = h_{21} w_1 + h_{22} w_2.$$

Then (6.2.10) implies that

$$h_{12} w_1 \wedge w_2 + h_{21} w_2 \wedge w_1 = 0.$$

So $h_{12} = h_{21}$. The shape operator is

$$-dN = -de_3 = -(w_{13} e_1 + w_{23} e_2) = w_{31} e_1 + w_{32} e_2 = \sum_{ij} h_{ij} w_i e_j.$$

Because e_1, e_2 are orthonormal, $h_{12} = h_{21}$ means that dN is self-adjoint. This gives a proof of Proposition 3.3.1 via differential forms.

By definition of II, we see that

$$\mathrm{II} = \sum_{ij} h_{ij} w_i \otimes w_j,$$

The mean and the Gaussian curvature are

$$H = h_{11} + h_{22}, \quad K = h_{11}h_{22} - h_{12}^2.$$

The first equation of the Gauss-Codazzi equation (6.2.3) is

$$\begin{aligned} dw_{12} &= -w_{13} \wedge w_{32} = w_{31} \wedge w_{32} = (h_{11}w_1 + h_{12}w_2) \wedge (h_{12}w_1 + h_{22}w_2) \\ &= (h_{11}h_{22} - h_{12}^2)w_1 \wedge w_2 = Kw_1 \wedge w_2. \end{aligned}$$

So we have

$$dw_{12} = Kw_1 \wedge w_2.$$

But we have proved that w_{12} can be computed from w_1, w_2. So K depends only on I. This gives another proof of the Gauss Theorem 5.2.1 using differential forms.

We summerize Cartan's moving frame method. Let M be an embedded surface in $\mathbb{R}^3$, e_1, e_2 be a local o.n. tangent frame on an open subset U of M, and w_1, w_2 the dual 1-form. Let e_3 be the unit normal field on M, and

$$w_{ij} = de_j \cdot e_i, \quad 1 \le i, j \le 3.$$

Then:

(1) The first and second fundametnal forms of M are

$$\begin{cases} \mathrm{I} = w_1^2 + w_2^2, \\ \mathrm{II} = w_1 \otimes w_{31} + w_2 \otimes w_{32}. \end{cases}$$

(2) w_1, w_2, w_{ij} satisfy the structure equation

$$\begin{cases} dw_1 = -w_{12} \wedge w_2, \\ dw_2 = -w_{21} \wedge w_1. \end{cases}, \tag{6.2.11}$$

and the Gauss and Codazzi equation

$$\begin{cases} dw_{12} = -w_{13} \wedge w_{32}, \\ dw_{13} = -w_{12} \wedge w_{23}, \\ dw_{23} = -w_{21} \wedge w_{23}. \end{cases} \tag{6.2.12}$$

(3) Write $w_{3i} = h_{i1}w_1 + h_{i2}w_2$. Then $h_{ij} = h_{ji}$, hence II is symmetric.

(4)

$$dw_{12} = -w_{13} \wedge w_{32} = Kw_1 \wedge w_2. \tag{6.2.13}$$

(5) w_{12} only depend on I and can be solved using the structure equation (6.2.11) and the Gaussian curvature K can be computed using I, namely

$$dw_{12} = Kw_1 \wedge w_2.$$

Next we state the Fundamental Theorem of surfaces in $\mathbb{R}^3$ in terms of differential forms. Given two symmetric bilinar forms

$$\mathrm{I} = \sum_{ij} g_{ij} dx_i dx_j, \quad \mathrm{II} = \sum_{ij} \ell_{ij} dx_i dx_j$$

on an open subset $\mathcal{O}$ of $\mathbb{R}^2$ such that I is positive definite. First, we complete the square for I to get an orthonormal dual 1-form:

$$\mathrm{I} = g_{11} \left(dx_1 + \frac{g_{12}}{g_{11}} dx_2 \right)^2 + \frac{g_{11}g_{22} - g_{12}^2}{g_{11}} dx_2^2.$$

Then $\mathrm{I} = w_1^2 + w_2^2$, where

$$w_1 = \sqrt{g_{11}} \left(dx_1 + \frac{g_{12}}{g_{11}} \, dx_2 \right), \quad w_2 = \sqrt{\frac{g_{11}g_{22} - g_{12}^2}{g_{11}}} dx_2.$$

Let w_{12} be the unique 1-form satisfies the structure equation (use the proof of Cartan's lemma to find the formula of w_{12}). Note that w_{31} and w_{32} can be solved algebraically by writing

$$\mathrm{II} = \sum_{ij} \ell_{ij} \, dx_i dx_j = w_1 \otimes w_{31} + w_2 \otimes w_{32}.$$

Suppose w_1, w_2, w_{ij} satisfy the structure (6.2.11) and Gauss-Codazzi equation (6.2.12). Then given $q_0 \in \mathcal{O}$, $p_0 \in \mathbb{R}^3$, and an o.n. basis u_1, u_2, u_3 of $\mathbb{R}^3$, we claim that there exists a unique solution (f, e_1, e_2, e_3) for

$$\begin{cases} df = w_1 e_1 + w_2 e_2, \\ de_i = \sum_{j=1}^3 w_{ji} e_j, & 1 \le i, j \le 3, \\ (f, e_1, e_2, e_3)(q_o) = (p_0, u_1, u_2, u_3). \end{cases} \tag{6.2.14}$$

with initial data (p_0, u_1, u_2, u_3). To prove the claim, we only need to check the compatibility condition. Note that solvability of the first equation of (6.2.14) is $d(w_1 e_1 + w_2 e_2) = 0$, which is the structure equation (6.2.9). The solvability of the second equation of (6.2.14) is the Gauss-Codazzi equation. This proves the claim.

We give an example using Cartan's moving frame method below:

Theorem 6.2.3. Hilbert Theorem for $K = -1$ surfaces in $\mathbb{R}^3$ *Suppose M is a surface in $\mathbb{R}^3$ with $K = -1$ given by an entire Tchebyshef asymptotic coordinate system (s, t), i.e., there is a coordinate system $f : \mathbb{R}^2 \to \mathbb{R}^3$ with $f(\mathbb{R}^2) = M$ so that the first and second fundamental forms are of the form* (6.1.6). *Then the area of M is less than 2π.*

Proof. We have proved that $\mathrm{I} = ds^2 + 2\cos 2q\ dsdt + dt^2$. Complete square to get an o.n. dual 1-forms:

$$\mathrm{I} = (ds + \cos 2q\ dt)^2 + (\sin\ 2qdt)^2.$$

Let

$$w_1 = ds + \cos 2q\ dt, \quad w_2 = \sin 2qdt.$$

We use the structure equation (6.2.11) to compute w_{12}. Let

$$w_{12} = Ads + Bdt.$$

Structure equation (6.2.11) implies that

$$\begin{cases} -2\sin 2q\ q_s = -A\sin 2q, \\ 2\cos 2q\ q_s = -B + A\cos 2q. \end{cases} \tag{6.2.15}$$

So $A = 2q_s$ and $B = 0$, i.e.,

$$w_{12} = 2q_s ds.$$

The Gauss equation is $dw_{12} = Kw_1 \wedge w_2$, so we have

$$dw_{12} = -w_1 \wedge w_2 = -\sin q \cos q\ ds \wedge dt. \tag{6.2.16}$$

The area of M is

$$\begin{aligned}
\mathrm{Area}(M) &= \int\int_{\mathbb{R}^2} w_1 \wedge w_2 = \int\int_{\mathbb{R}^2} (ds + \cos 2q\ dt) \wedge \sin 2q\ dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin 2q\ dsdt \quad \text{by (6.2.16)} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -dw_{12}, \quad \text{by the Green's formula (4.3.2)} \\
&= \lim_{r\to\infty} \int_{-r}^{r}\int_{-r}^{r} -w_{12} = -\lim_{r\to\infty} \oint 2q_s ds \\
&= -2\lim_{r\to\infty} \int_{-r}^{r} q_s(s,-r)ds + \int_{r}^{-r} q_s(s,r)ds \\
&= -2\lim_{r\to\infty} (q(r,-r) - q(-r,-r)) + (q(-r,r) - q(r,r)).
\end{aligned}$$

Since I is positive definite, $1 - \cos^2 2q = \sin^2 2q > 0$. We may assume $2q \in (0,\pi)$. So the above equation implies that the area of M is less than 2π. ☐

6.3. **Bäcklund Theorem.**

The idea of Bäcklund transformations comes from the study of line congruences. A *line congruence* in R^3 is a two-parameter family of lines

$$L(u,v): \quad x(u,v) + \tau\xi(u,v), \quad -\infty < \tau < \infty,$$

i.e., $L(u,v)$ is the line through $x(u,v)$ and parallel to $\xi(u,v)$. A surface M given by

$$Y(u,v) = x(u,v) + t(u,v)\xi(u,v)$$

for some smooth function t is called a *focal surface* of the line congruence if the line $L(u,v)$ is tangent to M at $Y(u,v)$ for all (u,v). Hence $\xi(u,v)$ lies in the tangent plane of M at $Y(u,v)$, which is spanned by

$$Y_u = x_u + t_u\xi + t\xi_u, \qquad Y_v = x_v + t_v\xi + t\xi_v.$$

Note that v_1, v_2, v_3 are linearly dependent if and only if $\det(v_1, v_2, v_3) = 0$. So t satisfies the following quadratic equation:

$$\det(\xi, x_u + t_u\xi + t\xi_u, x_v + t_v\xi + t\xi_v) = 0.$$

This implies

$$\begin{aligned} 0 &= \det(\xi, x_u + t\xi_u, x_v + t\xi_v) \\ &= t^2\det(\xi, \xi_u, \xi_v) + t(\det(\xi, \xi_u, x_v) + \det(\xi, x_u, \xi_v)) + \det(\xi, x_u, x_v) \end{aligned}$$

Note that x, ξ are given and the above equation is a quadratic equation in t. In general, this quadratic equation has two distinct solutions for t. Hence generically each line congruence has two focal surfaces, M and M^*. This results in a diffeomorphism $\ell: M \to M^*$ such that the line joining p and $p^* = \ell(p)$ is tangent to both M and M^*. We will call ℓ also a *line congruence*.

A line congruence $\ell: M \to M^*$ is called a *Bäcklund transformation* with constant θ if the angle between the normal of M at p and the normal of M^* at $p^* = \ell(p)$ is θ and the distance between p and p^* is $\sin\theta$ for all $p \in M$. We restate the definition below:

Definition 6.3.1. Let M, M^* be two surfaces in $\mathbb{R}^3$. A diffeomorphism $\ell: M \to M^*$ is called a *Bäcklund transformation with constant* θ if for any $P \in M$,

1. the line joing P and $P^* = \ell(P)$ is tangent to M at P and to M^* at P^* respectively,
2. the distance between P and P^* is $\sin\theta$,
3. the angle between the normal N_P of M at P and $N^*_{P^*}$ of M^* at P^* is equal to θ.

In 1883, A. Bäcklund proved:

Theorem 6.3.2. Bäcklund Theorem. *Let M and M^* be two surfaces in $\mathbb{R}^3$. If $\ell : M \to M^*$ is a Bäcklund transformation with constant θ, then both M and M^* have Gaussian curvature -1.*

Proof. Let e_1 denote the unit vector field on M so that

$$\ell(p) = p + \sin\theta\ e_1(p)$$

for all $p \in M$. Let e_2 be the unit vector field tangent to M that is perpendicular to e_1, and $e_3 = N$ the unit normal vector field on M. Then e_1, e_2, e_3 is a local o.n. frame on M. Let e_1^*, e_2^* be the tangent o.n. frame on M^* so that $e_1^* = -e_1$ (the direction of $\overrightarrow{\ell(p)p}$), and e_3^* is the unit normal of M^*. By assumption, the angle between e_3 and e_3^* is θ. So the two o.n. frames are related by

$$\begin{cases} e_1^* = -e_1, \\ e_2^* = \cos\theta e_2 + \sin\theta e_3, \\ e_3^* = -\sin\theta e_2 + \cos\theta e_3. \end{cases}$$

Let (w_{ij}) and (w_{ij}^*) be the $so(3)$-valued 1-form computed from

$$de_i = \sum_j w_{ji} e_j, \qquad de_i^* = \sum_j w_{ji}^* e_j^*. \tag{6.3.1}$$

Let $f : \mathcal{O} \to \mathbb{R}^3$ be a local parametrization of M, then

$$f^* = f + \sin\theta\ e_1$$

is a local parametrization of M^*. Let w_1, w_2 be the dual coframe of e_1, e_2, and w_1^*, w_2^* the dual coframe of e_1^*, e_2^*. So we have

$$df = w_1 e_1 + w_2 e_2, \qquad df^* = w_1^* e_1^* + w_2^* e_2^*. \tag{6.3.2}$$

The relation of these dual coframes can be computed by differentiating f and f^*:

$$\begin{aligned} df^* &= w_1^* e_1^* + w_2^* e_2^* \\ &= -w_1^* e_1 + w_2^*(\cos\theta e_2 + \sin\theta e_3) = d(f + \sin\theta\ e_1) \\ &= df + \sin\theta\ de_1 \qquad \text{by (6.3.2) and (6.3.1)} \\ &= w_1 e_1 + w_2 e_2 + \sin\theta\ (w_{21} e_2 + w_{31} e_3). \end{aligned}$$

Compare coefficients of e_1, e_2, e_3 of the above equation to get

$$w_1^* = -w_1, \tag{6.3.3}$$

$$w_2^* \cos\theta = w_2 + \sin\theta\ w_{21}, \tag{6.3.4}$$

$$w_2^* = w_{31}. \tag{6.3.5}$$

So we have

$$w_{31}\cos\theta = w_2 + \sin\theta\ w_{21}. \tag{6.3.6}$$

Use $de_i^* \cdot e_j^* = w_{ij}^*$ to compute

(6.3.7)

$$w_{31}^* = de_1^* \cdot e_3^* = -de_1\cdot(-\sin\theta e_2 + \cos\theta e_3) = \sin\theta w_{21} - \cos\theta w_{31} = -w_2.$$

Here we use (6.3.6). We also have

$$\begin{aligned} w_{32}^* &= de_2^* \cdot e_3^* = (\cos\theta de_2 + \sin\theta de_3)\cdot(-\sin\theta e_2 + \cos\theta e_3) \\ &= \cos^2\theta w_{32} - \sin^2\theta w_{23} = w_{32}. \end{aligned} \tag{6.3.8}$$

Let $w_{3i} = \sum_j h_{ij} w_j$, and K, K^* the Gaussian curvature of M and M^* respectively. Use (6.2.13), (6.3.7) and (6.3.8) to get

$$\begin{aligned} K^* w_1^* \wedge w_2^* = w_{31}^* \wedge w_{32}^* &= -w_2 \wedge w_{32} = h_{12} w_1 \wedge w_2 \\ &= -K^* w_1 \wedge w_{31} = -K^* h_{12} w_1 \wedge w_2. \end{aligned}$$

This implies that $-K^* h_{12} = h_{12}$. Hence $K^* = -1$. Since the situation is symmetric, $K = -1$ too. □

Theorem 6.3.3. *Given a surface M in $\mathbb{R}^3$, a constant $0 < \theta < \pi$, and a unit vector $v_0 \in TM_{p_0}$ not a principal direction, then there exists a unique surface M^* and a Bäcklund transformation $\ell : M \to M^*$ with constant θ such that $\overrightarrow{p_0\ell(p_0)} = (\sin\theta)v_0$.*

Proof. To construct a Bäcklund transformation ℓ with constant θ is the same as to find a unit vector field e_1 on M so that the corresponding frames w_i and (w_{ij}) satisfy the equation (6.3.6). We can rewrite this differential form equation as a system of first order PDE as follows. Let v_1, v_2, v_3 be a local o.n. frame on M so that v_1, v_2 are principal directions. Let (x_1, x_2) be the local line of curvature coordinates constructed in Theorem 6.1.1, and $q(x_1, x_2)$ the corresponding solution of the SGE (6.1.3). We know from the proof of Theorem 6.1.1 that

$$\theta_1 = \cos q\ dx_1, \quad \theta_2 = \sin q\ dx_2$$

are the 1-forms dual to v_1, v_2, and $\theta_{ij} = dv_j \cdot v_i = p_{ij}dx_1 + q_{ij}dx_2$ is given as follows:

$$\begin{aligned} \theta_{12} &= -q_{x_2}dx_1 - q_{x_1}dx_2, \\ \theta_{13} &= -\sin q\ dx_1, \\ \theta_{23} &= \cos q\ dx_2. \end{aligned}$$

Let q^* denote the angle between v_1 and e_1, and e_1, e_2, e_3 an o.n. frame on M so that $e_3 = v_3$, i.e.,

$$\begin{cases} e_1 = \cos q^* \ v_1 + \sin q^* \ v_2, \\ e_2 = -\sin q^* \ v_1 + \cos q^* \ v_2, \\ e_3 = v_3. \end{cases} \tag{6.3.9}$$

Let $w_{ij} = de_j \cdot e_i$. Use (6.3.9) to get

$$\begin{aligned} w_{31} &= de_1 \cdot e_3 = (\cos q^* \ dv_1 + \sin q^* \ dv_2) \cdot v_3 = \cos q^* \ \theta_{31} + \sin q^* \ \theta_{32} \\ &= \cos q^* \ \sin q \ dx_1 - \sin q^* \ \cos q \ dx_2, \\ w_2 &= -\sin q^* \ \theta_1 + \cos q^* \ \theta_2 = -\sin q^* \cos q \ dx_1 + \cos q^* \ \sin q \ dx_2, \\ w_{21} &= de_1 \cdot e_2 = dq^* + \theta_{21} = dq^* + q_{x_2} \ dx_1 + q_{x_1} \ dx_2. \end{aligned}$$

Equate the coefficients of dx_1 and dx_2 of equation (6.3.6) to get

$$\begin{cases} -\sin q^* \cos q + \sin\theta((q^*)_{x_1} + q_{x_2}) = \cos\theta \cos q^* \sin q, \\ \cos q^* \sin q + \sin\theta((q^*)_{x_2} + q_{x_1}) = -\cos\theta \sin q^* \cos q. \end{cases} \tag{6.3.10}$$

Now we make a change of coordinates:

$$x_1 = s + t, \quad x_2 = s - t,$$

i.e., (s, t) is the Tchebyshef asymptotic coordinate system. Note that

$$u_s = u_{x_1} + u_{x_2}, \quad u_t = u_{x_1} - u_{x_2}.$$

Add and substract the two equations of (6.3.10) and use (s, t) coordinates, we see that (6.3.10) gives

$$\begin{cases} \sin\theta \ (q^* + q)_s = (1 - \cos\theta)\sin(q^* - q), \\ \sin\theta \ (q^* - q)_t = (1 + \cos\theta)\sin(q^* + q). \end{cases}$$

So we have

$$\begin{cases} (q^* + q)_s = \frac{(1-\cos\theta)}{\sin\theta}\sin(q^* - q), \\ (q^* - q)_t = \frac{(1+\cos\theta)}{\sin\theta}\sin(q^* + q) \end{cases}$$

Let $\mu = \frac{(1-\cos\theta)}{\sin\theta}$. Then $\frac{1}{\mu} = \frac{1+\cos\theta}{\sin\theta}$. So the above system becomes

$$\begin{cases} (q^* + q)_s = \mu \sin(q^* - q), \\ (q^* - q)_t = \frac{1}{\mu}\sin(q^* + q), \end{cases} \tag{6.3.11}$$

i.e.,

$$\begin{cases} (q^*)_s = -q_s + \mu \sin(q^* - q), \\ (q^*)_t = q_t + \frac{1}{\mu}\sin(q^* + q). \end{cases} \tag{6.3.12}$$

To find the unit vector field e_1 so that $\ell = f + \sin\theta e_1$ is a Bäcklund transformation is equivalent to solve the differential form equation (6.3.6), which is equivalent to solve the PDE system (6.3.12).

By the Frobenius Theorem 2.4.1, the condition for system (6.3.12) to be solvable is to equate the t-derivative of the first equation of (6.3.12) and the s-derivative of the second equation of (6.3.12):

$$\begin{aligned}((q^*)_s)_t &= -q_{st} + \mu\cos(q^* - q)\ (q^* - q)_t \\ &= -q_{st} + \cos(q^* - q)\ \sin(q^* + q), \\ = ((q^*)_t)_s &= q_{st} + \frac{1}{\mu}\cos(q^* + q)\ (q^* + q)_s \\ &= q_{st} + \cos(q^* + q)\ \sin(q^* - q).\end{aligned}$$

So the compatibility condition is

$$2q_{st} = \cos(q^* - q)\ \sin(q^* + q) - \cos(q^* + q)\ \sin(q^* - q) = \sin(2q),$$

i.e., q satisfies the SGE (6.1.7). □

We use $\mathrm{BT}_{q,\mu}$ to denote the system (6.3.11) (or (6.3.12)) for given q and constant μ. The proof of the above theorem gives a result in PDE.

Corollary 6.3.4. *Let μ be a non-zero real constant and $q(s,t)$ a smooth function. Then the system* (6.3.11) *for q^* is solvable if and only if q satisfies the SGE* (6.1.7)*, i.e., $2q_{st} = \sin(2q)$. Moreover, if q^* is a solution of* (6.3.11)*, then q^* is also a solution of the SGE* (6.1.7)*.*

The above Corollary also implies that if q is a solution of the SGE and a real constant μ, then the first order system $\mathrm{BT}_{q,\mu}$ (6.3.11) is solvable for q^* and the solution q^* is again a solution of the SGE. So we can solve the system $\mathrm{BT}_{q,\mu}$ of two compatible ODE to get a family of solutions of SGE. If we apply this method again, we get a second family of solutions. This gives an inifintely many families of solutions from one given solution of SGE.

For example, the constant function $q = 0$ is a trivial solution of the SGE. The system $\mathrm{BT}_{0,\mu}$ is

$$\begin{cases}\alpha_s = \mu\sin\alpha, \\ \alpha_t = \frac{1}{\mu}\sin\alpha.\end{cases}$$

It has explicit solution

$$\alpha(s,t) = 2\tan^{-1}\left(e^{\mu s + \frac{1}{\mu}\ t}\right).$$

We can apply Bäcklund transformation $\mathrm{BT}_{\alpha,\mu_1}$ to this new solution α to get another family of solutions. Note that this two system of ODEs are not as easy to solve as $\mathrm{BT}_{0.\mu}$. But instead of solving $\mathrm{BT}_{\alpha,\mu_1}$ we can use the following classical Bianchi's theorem, which says that the second family of solutions can be constructed from the first family of solutions by an explicit algebraic formula:

Theorem 6.3.5. Bianchi Permutability Theorem.
Let $0 < \theta_1, \theta_2 < \pi$ be constants so that $\sin\theta_1^2 \neq \sin\theta_2^2$, and $\ell_i : M_0 \to M_i$ Bäcklund transformations with constant θ_i for $i = 1, 2$. Then there exists a unique surface M_3 and Bäcklund transformations $\tilde{\ell}_1 : M_2 \to M_3$ and $\tilde{\ell}_2 : M_1 \to M_3$ with constant θ_1, θ_2 respectively so that $\tilde{\ell}_1 \circ \ell_2 = \tilde{\ell}_2 \circ \ell_1$. Moreover, if q_i is the solution of the SGE corresponding to M_i for $0 \le i \le 3$, then

$$\tan\left(\frac{q_3 - q_0}{2}\right) = \frac{\mu_1 + \mu_2}{\mu_1 - \mu_2}\tan\left(\frac{q_1 - q_2}{2}\right), \tag{6.3.13}$$

where $\mu_i = \frac{1-\cos\theta_i}{\sin\theta_i}$.

The proof of the geometric part of the above theorem is rather long and tidious (cf. []). But to prove that the function q_3 defined by (6.3.13) solves the system BT_{q_1,μ_2} and BT_{q_2,μ_1} can be done by a direct computation. Equation (6.3.13) gives a formula of q_3 in terms of q_0, q_1, q_2.

When $q_0 = 0$, we have solved $q_i = 2\tan^{-1} e^{\mu_i s + \frac{1}{\mu_i} t}$ for $i = 1, 2$. So apply (6.3.13) get a second family of solutions:

$$\tan\frac{q_{12}}{2} = \frac{\mu_1 + \mu_2}{\mu_1 - \mu_2}\left(\frac{e^{\mu_1 s + \frac{1}{\mu_1} t} - e^{\mu_2 s + \frac{1}{\mu_2} t}}{1 + e^{(\mu_1 + \mu_2)s + (\frac{1}{\mu_1} + \frac{1}{\mu_2})t}}\right). \tag{6.3.14}$$

Apply (6.3.13) again, we get a third family of solutions. To be more precise, let μ_1, μ_2, μ_3 be real numbers so that $\mu_1^2, \mu_2^2, \mu_3^2$ are distinct, q_{12} the solution defined by (6.3.14), and q_{23} the solution given by the same formula (6.3.14) replacing μ_1, μ_2 by μ_2, μ_3. Then

$$\tan\frac{q_{123}}{2} = \frac{\mu_1 + \mu_3}{\mu_1 - \mu_3}\tan\left(\frac{q_{12} - q_{23}}{2}\right).$$

We can continue this process to construct an infinitely many families of explicit solutions of the SGE.

Department of Mathematics, Northeastern University, Boston, MA 02115

E-mail address: `terng@neu.eud`