Pseudosphere

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The Pseudosphere is a surface of revolution (of the Tractrix) and has Gaussian curvature minus one, or in other words, the product of its principal curvatures is -1. On a surface of revolution, this translates into a simple analytic property: Parametrize the meridian curve by arc length $s \to (\mathrm{r}(s), \mathrm{h}(s)), r'^2 + h'^2 = 1$. Then r is a solution of the differential equation $r'' = r$, and consequently h is also known—it is the anti-derivative of $\sqrt{1 - r'^2}$.

The Pseudosphere is best known because its intrinsic geometry is hyperbolic, the meridians are a family of asymptotic geodesics and the orthogonal latitudes are therefore a geodesically parallel family of "horocycles", i.e. limits of circles as their midpoints converge to the limit point of the asymptotic geodesics.

This Pseudosphere is obtained by the construction which relates solutions of the Sine-Gordon equation to surfaces of Gaussian curvature -1, here the solution is a one-soliton solution:

$q(u, v) := 4 \arctan(\exp(u))$.

The parametrization obtained has another remarkable property: The diagonal curves in *all* the parameter quadrilaterals have the same length! Nets used for fishing also have such equiquadrilaterals as meshes; the mathematical term is "Tchebycheff net". Such Tchebycheff nets exist on all surfaces which are isometric immersions of (portions of) the hyperbolic plane. This fact plays a key role in the proof of Hilbert's theorem which says: There is no smooth isometric immersion of the whole hyperbolic plane into euclidean threespace.

See also the "Tractrix" under Planar Curves.